Stability and Convergence of Relaxation Schemes to Hyperbolic Balance Laws via a Wave Operator

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Alexey Miroshnikov* and Konstantina Trivisa †

Abstract

This article deals with relaxation approximations of nonlinear systems of hyperbolic balance laws. We introduce a class of relaxation schemes and establish their stability and convergence to the solution of hyperbolic balance laws before the formation of shocks, provided that we are within the framework of the compensated compactness method. Our analysis treats systems of hyperbolic balance laws with source terms satisfying a special mechanism which induces weak dissipation in the spirit of Dafermos [11], as well as hyperbolic balance laws with more general source terms. The rate of convergence of the relaxation system to a solution of the balance laws in the smooth regime is established. Our work follows in spirit the analysis presented in [1, 14] for systems of hyperbolic conservation laws without source terms.

1 Introduction

Relaxation approximations of hyperbolic balance laws is of essence for the investigation of models arising in continuum mechanics and kinetic theory of gases, and serve as a ground stage for the design of numerical schemes for hyperbolic balance laws. In this article we introduce a class of relaxation schemes for the approximation of solutions to the hyperbolic balance law

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = G(u), \quad u \in \mathbb{R}^n,\ (x,t) \in \mathbb{R}^d \times [0,\infty) \tag{1.1}$$

and address the issues of stability and convergence.

The class of relaxation schemes introduced in this work are of the form

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{d} \partial_{x_j} v_j = 0 \\ \partial_t v_i + A_i \partial_{x_i} u = -\dfrac{1}{\varepsilon}\Big(v_i - F_i(u) + \mathcal{R}_i(x,t)\Big), \quad i = 1,\dots,d \end{cases} \tag{1.2}$$

*Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA
†Department of Mathematics, University of Maryland, College Park, MD 20742, USA

with $v_i \in \mathbb{R}^n$, A_i symmetric, positive definite matrix, and

$$\mathcal{R}_i(x,t) = \frac{1}{d}\int^{x_i} G(u(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_d,t))\,dz. \tag{1.3}$$

Excluding v_i for all $i=1,\dots,d$ from the equation $(1.2)_1$ we obtain

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = G(u) + \varepsilon\Big(\sum_{j=1}^{d} A_j u_{x_j x_j} - u_{tt}\Big) \tag{1.4}$$

that approximates the system of balance laws (1.1). For the relaxation approximations considered in this work the stabilization mechanism is the regularization by the *wave operator* in (1.4).

The convergence properties of relaxation systems and associated relaxation schemes for scalar conservation laws are presently well understood (see [2, 8, 14, 17]). When the zero-relaxation limit is a system of conservation laws, the dissipative effect of relaxation is often subtle to capture, yet there is a vast literature on convergence results in that direction (see [1, 23, 21, 28, 29] and the references there rein).

By contrast, the relaxation approximation of nonlinear systems of hyperbolic balance laws presents major additional challenges and it is the subject of current intense research investigation. It is well known that standard methods that solve correctly systems of conservation laws can fail in solving systems of balance laws, especially when approaching equilibria or near to equilibria solutions. In addition, standard approximating procedures produce often unstable methods when they are applied to coupled systems of conservation or balance laws. Even at the theoretical level, the presence of the production term (source term) in the equation results to the amplification in time of even small oscillations in the solution. Due to these challenges special mechanisms that induce dissipation are often desirable and have been proven effective in obtaining long-term stability (cf. Dafermos [11], [12]).

In the present article we identify a class of relaxation schemes suitable for the approximation of solutions to certain systems of hyperbolic balance laws arising in continuum physics. The relaxation schemes proposed in our work provide a very effective mechanism for the approximation of the solutions of these systems with a very high degree of accuracy.

The main contribution of the present article to the existing theory can be characterized as follows:

- A new class of relaxation schemes (1.2) is introduced. The novelty of the relaxation systems proposed in this work lies in the introduction of the *global term* $\{\mathcal{R}_i(x,t)\}$ in (1.2)-(1.3). The presence of this term in the relaxation system allows us to relax both the flux and the antiderivative of the source term simultaneously. We refer the reader to the article [19], where a relevant idea was proposed for the numerical treatment of shallow water equations. The present article is the first step towards the construction of fully discrete schemes and the development of numerical methods for the approximation of solutions to complex nonlinear multidimensional systems of hyperbolic balance laws arising in applications. Our analysis provides a rigorous proof of the relaxation limit and a rate of convergence before the formation of shocks.

- A comparison is presented between the relaxation system introduced here and an alternative relaxation system for which the source term $G(u)$ appears in the right-hand side of the first equation (see (7.1) in Section 6). Note, dealing with (7.1) one faces arduous challenges. More specifically, the time derivative of the source term appears in the energy functional posing enormous challenges in the analysis, an additional hypothesis is required for the establishment of stability (see (H7), Section 6), the issue of compactness is problematic.

- The presence of a source term in our system requires us to modify the relative entropy method significantly. The *modified relative method* presented in this work relies on a *relative potential* (Section 8). The introduction of this concept is required in order to deal with the source term G which typically satisfies no growth conditions. The relative potential assists in "tracking" the contribution of the source term; it becomes a part of a Lyapunov functional which monitors the evolution of the difference of the solutions and enables us to establish a convergence rate of order $\mathcal{O}(\varepsilon^2)$. In the case of the general source G the terms associated with the source are treated as error (cf. Section 6.4).

The reader should contrast the relaxation system presented in this work with other relaxation systems proposed in the literature [16, 17], where relaxation approximations to hyperbolic balance laws were proposed and rigorously established. In [16] relaxation approximations are constructed by the introduction of special variables the so-called *internal variables.* In that setting applications to physical systems in elasticity and combustion theory are presented. In the former case, relaxation is introduced via *stress approximation*, whereas in the latter case via *approximation of pressure.* Reference [17] treats the Cauchy problem for 2×2 semilinear and quasilinear hyperbolic systems with a singular relaxation term and presents the convergence to equilibrium of the solutions of these problems as the singular perturbation parameter tends to 0.

In the center of our analysis lies the entropy structure of the balance law and the dissipative nature of the source term. The main ingredients of our approach can be formulated as follows:

- The representation of the *global term* $\mathcal{R}$ in the formulation of the relaxation system enables us to obtain the stability estimates in Section 3. These estimates are used subsequently to establish the stability of the relaxation approximations and in fact justify the dissipative character of our systems.

- The entropy structure of the balance law provides the basis for the use of the compensated compactness method. Recall that a pair of functions $\eta = \eta(u), q = q(u)$ are called the entropy-entropy flux pair if (η, q) solve the linear hyperbolic system

$$\mathrm{D}q = \mathrm{D}\eta \mathrm{D}F.$$

  In Section 5, we show that the relaxation approximations satisfy

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \subset \text{compact set of } H^{-1}_{loc}(\mathcal{O})$$

  for a certain class of entropy-entropy flux pairs $\eta - q$, which is one of the main ingredients for the establishment of convergence results within the compensated compactness method (cf. Serre [24]).

- The Lyapunov functional construction in Section 8 relies on the relative entropy via a Chapman-Enskog-type expansion and is used to provide a simple and direct convergence framework before formation of shocks as well as suitable error estimates.

- A physically motivated dissipation mechanism associated with the source term in (1.1). The concept of *weak dissipation* for hyperbolic balance laws was introduced by Dafermos in [11]. The reader may contrast the result of Theorem 5.3.1 for weakly dissipative source terms with Theorem 5.4.1 which corresponds to the case of a general source.

The outline of our article is as follows: In Section 2 we present the basic notation and hypothesis. In Section 3 we present the stability estimates which yield the stability of the relaxation systems. The compactness properties of the approximate solutions are discussed in Section 4. Section 5 is devoted to error estimates for smooth solutions via the relative entropy method as well as proofs of convergence. The multidimensional case is treated in Section 7. Section 8 presents applications in elasticity and combustion.

2 Notation and Hypotheses

For the convenience of the reader we collect in this section all the relevant notation and hypotheses. Here and in what follows:

1. $G, R, F_i,\ i = 1, \ldots, d$ denote the mappings $G, R, F_i : \mathbb{R}^n \to \mathbb{R}^n$. In our presentation, $G(u), R(u), F_i(u)$ are treated as column vectors.

2. D denotes the differential with respect to the state vectors $u \in \mathbb{R}^n$. When used in conjunction with matrix notation, D represents a row operation:

$$\mathrm{D} = [\partial/\partial u^1, \ldots, \partial/\partial u^n].$$

2.1 Entropy Structure

Some additional assumptions on the system (1.1) read:

- The system (1.1) is equipped with a globally defined entropy $\eta(u)$ and corresponding fluxes $q_i(u)$, $i = 1, \ldots, d$, such that

$$\begin{aligned} &\eta : \mathbb{R}^n \to \mathbb{R} \ \text{ is strictly convex,} \\ &\mathrm{D}\eta(u)\mathrm{D}F_i(u) = \mathrm{D}q_i(u) \\ &\beta\mathbf{I} \leq D^2\eta(u) \leq \tfrac{1}{2}\alpha\mathbf{I}, \quad u \in \mathbb{R}^n, \ \text{for } \alpha, \beta > 0, \\ &\eta(u) \geq \eta(0) = 0, \, \mathrm{D}\eta(0) = 0. \end{aligned} \tag{H1}$$

We recall that 2×2 as well as physical systems of hyperbolic conservation laws are always equipped with an entropy-entropy flux pair. The same holds true for symmetric hyperbolic systems [12].

2.2 Subcharacteristic condition

The Whitham relaxation *subcharacteristic condition* presented below will be essential for the dissipativiness of our system [2, 12, 15, 17].

- (Case $d = 1$. Systems with a strictly convect entropy).

  For $\alpha > 0$ in (H1) there exists $\nu > 0$ and symmetric, positive definite matrix A such that

$$\tfrac{1}{2}\big(A\mathrm{D}^2\eta(u) + \mathrm{D}^2\eta(u)A\big) - \alpha \mathrm{D}F(u)^\top \mathrm{D}F(u) \geq \nu \mathbf{I}\,, \quad u \in \mathbb{R}^n. \tag{H2}$$

  The positivity of this term is required for dissipativiness in Lemma 3.1.1.

- (General case $d \geq 1$. Systems with a strictly convect entropy).

  For $\alpha > 0$ in (H1) there exists $\nu > 0$ and symmetric, positive definite matrices A_j, $j = 1, \ldots, d$ such that

$$\tfrac{1}{2}\,\xi_j^\top \big(A_j\mathrm{D}^2\eta(u) + \mathrm{D}^2\eta(u)A_i\big)\xi_j - \alpha\left|\sum_{i=1}^{d} \mathrm{D}F(u)^\top \xi_j\right|^2 \geq \nu \sum_{i=1}^{d} |\xi_j|\,, \quad \forall \xi_1, \ldots, \xi_d \in \mathbb{R}^n,\ \ u \in \mathbb{R}^n. \tag{H2*}$$

2.3 Dissipation

The following hypotheses will be relevant to our subsequent discussion.

- The source term $G(u)$ is *weakly dissipative* in the sense of Definition 2.3.1.

  Definition 2.3.1. We say that the source $G(u)$ is *weakly dissipative*, if

$$-\big(\mathrm{D}\eta(u) - \mathrm{D}\eta(\bar{u})\big)\big(G(u) - G(\bar{u})\big) \geq 0\,, \quad u, \bar{u} \in \mathbb{R}^n. \tag{H3-a}$$

An alternative condition on the source G, exploited in Theorem 5.4.1, reads:

- Suppose that for every compact set $\mathcal{A}$ there exists $L_{\mathcal{A}} > 0$ such that

$$|G(u) - G(\bar{u})| \leq L_{\mathcal{A}}|u - \bar{u}|\,, \quad u \in \mathbb{R}^n,\ \bar{u} \in \mathcal{A}\,. \tag{H3-b}$$

Through out the article we use the assumptions that $G(0) = 0$ and $G \in C(\mathbb{R}^n)$.

2.4 Source potential

For the weakly dissipative source G we employ an additional assumption that it is a gradient:

- Suppose there exists a potential $R(u) : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &G(u) = -\mathrm{D}R(u)^\top \\ &R(u) \geq R(0) = 0,\ \mathrm{D}R(0)^\top = 0 \\ &|\mathrm{D}R(u)| = |G(u)| \ \leq\ C_R\big(1 + R(u)\big), \quad u \in \mathbb{R}^n. \end{aligned} \tag{H4}$$

3 Stability estimates

3.1 Systems with a strictly convex entropy η, $d = 1$.

The balance law (1.1) for $d = 1$ reads

$$\partial_t u + \partial_x F(u) = G(u), \quad u \in \mathbb{R}^n \tag{3.1}$$

and the relaxation model by

$$\begin{cases} \partial_t u + \partial_x v = 0 \\ \partial_t v + A\partial_x u = -\dfrac{1}{\varepsilon}\Big(v - F(u) + \displaystyle\int^x G\big(u(x,t)\big)\,dx\Big) \end{cases} \tag{3.2}$$

with A symmetric, positive definite matrix. In that case the second order relaxation system (1.4) reads

$$\partial_t u + \partial_x F(u) = G(u) + \varepsilon(Au_{xx} - u_{tt}). \tag{3.3}$$

We now consider the hyperbolic system (3.1) that is equipped with the entropy-entropy flux pair $\eta - q$, with η strictly convex, and establish stability results for the relaxation model (3.3).

Lemma 3.1.1. *Suppose $u \equiv u^\varepsilon(x,t)$ is a smooth solution to the equation (3.3) on $\mathbb{R} \times [0,T]$, $\eta - q$ is the entropy-entropy flux pair of (3.1) and $\bar{\alpha} \in \mathbb{R}$ is fixed. Then, the following energy identity holds*

$$\begin{aligned} &\partial_t\Big[\eta(u+\varepsilon u_t) + \tfrac{1}{2}\varepsilon^2\bar{\alpha}\,|u_t|^2 + \varepsilon^2\bar{\alpha}\,u_x^\top A u_x \\ &\qquad + \varepsilon^2 u_t^\top\Big(\tfrac{1}{2}\bar{\alpha}\mathbf{I} - \int_0^1\!\!\int_0^s \mathrm{D}^2\eta(u+\varepsilon\tau u_t)\,d\tau ds\Big)u_t\Big] + \partial_x q(u) \\ &\quad + \varepsilon\bar{\alpha}\big|u_t + \mathrm{D}F(u)u_x\big|^2 + \varepsilon u_t^\top\Big(\bar{\alpha}I - \mathrm{D}^2\eta(u)\Big)u_t \\ &\quad + \varepsilon u_x^\top\Big(\mathrm{D}^2\eta(u)A - \bar{\alpha}\mathrm{D}F(u)^\top\mathrm{D}F(u)\Big)u_x \\ &= \partial_x\Big(\varepsilon\mathrm{D}\eta(u)Au_x + 2\varepsilon^2\bar{\alpha}u_t^\top Au_x\Big) + \mathrm{D}\eta(u)G(u) + 2\varepsilon\bar{\alpha}u_t^\top G(u). \end{aligned} \tag{3.4}$$

Proof. Multiplying (3.3) by $u_t^\top$ we obtain

$$\begin{aligned} &\partial_t\Big(\tfrac{1}{2}\varepsilon|u_t|^2 + \tfrac{1}{2}\varepsilon u_x^\top Au_x\Big) + |u_t|^2 + u_t^\top\mathrm{D}F(u)u_x \\ &= \partial_x\Big(\varepsilon u_t^\top Au_x\Big) + u_t^\top G(u). \end{aligned} \tag{3.5}$$

Similarly, multiplying (3.3) by $\mathrm{D}\eta(u)$ we get

$$\begin{aligned} &\partial_t\Big(\eta(u) + \varepsilon\mathrm{D}\eta(u)u_t\Big) + \partial_x q(u) \\ &+ \varepsilon\Big(\big(\mathrm{D}^2\eta(u)u_x\big)^\top Au_x - u_t^\top\mathrm{D}^2\eta(u)u_t\Big) \\ &= \partial_x\Big(\varepsilon\mathrm{D}\eta(u)Au_x\Big) + \mathrm{D}\eta(u)G(u). \end{aligned} \tag{3.6}$$

Now, we multiply (3.5) by $2\bar{\alpha}\varepsilon$, add (3.6) and use the identity

$$\eta(u+\varepsilon u_t) = \eta(u) + \varepsilon \mathrm{D}\eta(u) u_t + \varepsilon^2 u_t^\top \left(\int_0^1 \int_0^s \mathrm{D}^2 \eta(u+\varepsilon\tau u_t)\, d\tau\, ds \right) u_t \tag{3.7}$$

to deduce (3.4). This proves the lemma. □

We now establish the stability of solutions $\{u^\varepsilon\}$. For that we will require the matrix A to satisfy the subcharacteristic condition (H2) and make use of hypotheses (H3-a)-(H4) to control the source G. In the sequel we will use the notation

$$\varphi(t) := \int_{\mathbb{R}} |u^\varepsilon|^2 + \varepsilon^2 |u_x^\varepsilon|^2 + \varepsilon^2 |u_t^\varepsilon|^2\, dx \tag{3.8}$$

with u^ε denoting a solution of (3.3).

Proposition 3.1.2 (**Weakly dissipative source**). *Let $\{u^\varepsilon(x,t)\}$ be a family of smooth solutions to the equation* (3.3) *on $\mathbb{R}\times[0,T]$. Suppose that $u \equiv u^\varepsilon$ decays fast at infinity and that:*

(a1) (H1) *holds true, and the positive definite, symmetric matrix A is such that the subcharacteristic condition* (H2) *is valid.*

(a2) The conditions (H3-a) *and* (H4) *for the source G hold true.*

Then for all $t\in[0,T]$

$$\varphi(t) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,t))\, dx + \int_0^t \int_{\mathbb{R}} |\mathrm{D}\eta(u) G(u)|\, dxdt \;\le\; C\Big(\varphi(0) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,0))\, dx\Big) \tag{3.9}$$

with φ defined in (3.8) *and $C = C(A,\alpha,\beta) > 0$ independent of both ε and T.*

Proof. By (H1) we have

$$0 \le u_t^\top \Big(\tfrac{1}{2}\alpha \mathbf{I} - \int_0^1 \int_0^s \mathrm{D}^2 \eta(u+\varepsilon\tau u_t)\, d\tau ds\Big) u_t \;\le\; \tfrac{1}{2}\alpha |u_t|^2. \tag{3.10}$$

Then, integrating the identity (3.4), with $\bar{\alpha} = \alpha$, and using hypotheses (H3-a), (H4) we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \Big(\eta(u+\varepsilon u_t) + \tfrac{1}{2}\varepsilon^2 \alpha |u_t|^2 + \varepsilon^2 \alpha u_x^\top A u_x + 2\varepsilon\alpha R(u)\Big)\, dxdt \\ &\quad + \int_0^t \int_{\mathbb{R}} \varepsilon\nu |u_x|^2 + \tfrac{1}{2}\varepsilon\alpha |u_t|^2\, dxdt \\ &\quad + \int_0^t \int_{\mathbb{R}} \varepsilon\alpha |u_t + \mathrm{D}F(u) u_x|^2 + |\mathrm{D}\eta(u) G(u)|\, dxdt \\ &\quad \le \int_{\mathbb{R}} \Big(\eta(u_0 + \varepsilon u_{0t}) + c\varepsilon^2 \alpha |u_{0t}|^2 + \varepsilon^2 u_{0x}^\top A u_{0x} + 2\varepsilon\alpha R(u_0)\Big)\, dx \end{aligned} \tag{3.11}$$

with $\frac{1}{2} \le c \le 1$. From (H1) it follows that

$$c_1 \varphi(t) \;<\; \int_{\mathbb{R}} \Big(\eta(u+\varepsilon u_t) + \tfrac{1}{2}\alpha\varepsilon^2 |u_t|^2 + \varepsilon^2 \alpha u_x^\top A u_x\Big)\, dx < c_2 \varphi(t) \tag{3.12}$$

for some c_1, c_2 that depend on α, β and A. Then, combining (3.11) and (3.12), we obtain (3.9). □

Proposition 3.1.3 (General source). *Let $\{u^\varepsilon(x,t)\}$ be a family of smooth solutions to the equation* (3.3) *on $\mathbb{R}\times[0,T]$. Suppose that $u\equiv u^\varepsilon$ decays fast at infinity and that:*

(a1) (H1) *holds true, and the positive definite, symmetric matrix A, is such that the subcharacteristic condition* (H2) *is valid.*

(a2) The condition (H3-b) *for the source G holds true.*

Then,

$$\varphi(t)\leq C\,\varphi(0)\,,\quad t\in[0,T] \tag{3.13}$$

with φ defined in (3.8) *and $C=C(A,\alpha,\beta,T,L)>0$ independent of ε.*

Proof. Integrating the energy identity (3.4), with $\bar\alpha=\alpha$, and using (H1), (H2) together with relations (3.10), (3.12) we obtain for $\tau\in[0,T]$

$$c_1\varphi(\tau)\leq c_2\,\varphi(0)+\int_0^\tau\int_{\mathbb{R}}\Big(\mathrm{D}\eta(u)G(u)+2\varepsilon\alpha u_t^\top G(u)\Big)dxdt\,. \tag{3.14}$$

Since $G(0)=0$, (H3-b) implies $|G(u)|\leq L|u|$ and therefore, in view of (H1),

$$|D\eta(u)G(u)+2\varepsilon\alpha u_t^\top G(u)|\leq\alpha L|u|^2+\alpha\varepsilon^2|u_t|^2+\alpha L^2|u|^2\,.$$

Then (3.8) and (3.14) imply

$$\varphi(\tau)\leq c\Big(\varphi(0)+\int_0^\tau\varphi(t)\,dt\Big)$$

with $c>0$ depending on c_1,c_2 and L. Then, we conclude (3.13) via the Gronwall lemma. □

4 Compactness properties

This section focuses on systems of hyperbolic balance laws with a strictly convex entropy, $d=1$.

4.1 Systems with a strictly convex entropy, $d=1$.

Starting with a family $\{u_\varepsilon\}$ smooth solutions of (3.3) on $\mathbb{R}\times[0,\infty)$ the goal in this section is to control the dissipation measure and to show that

$$\partial_t\bar\eta(u^\varepsilon)+\partial_x\bar q(u^\varepsilon)\text{ lies in a compact set of }\;H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$$

for a certain class of entropy-entropy flux pairs $(\bar\eta,\bar q)$. In the proof we use Murat's lemma [22].

Lemma 4.1.1 (Murat's Lemma [22]). *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^m$ and $\{\phi_j\}$ a bounded sequence of $W^{-1,p}(\mathcal{O})$ for some $p>2$. In addition let $\phi_j=\chi_j+\psi_j$, where $\{\chi_j\}$ belongs in a compact set of $H^{-1}(\mathcal{O})$ and $\{\psi_j\}$ belongs in a bounded set of the space of measures $M(\mathcal{O})$. Then $\{\phi_j\}$ belongs in a compact set of $H^{-1}(\mathcal{O})$.*

In the presence of uniform L^∞-bounds the compensated compactness framework (cf. Tartar [27], DiPerna [13]) guarantees compactness of approximate solutions and implies that, along a subsequence, $u_\varepsilon \to u$ *a.e.* (x,t). In the absence of L^∞-bounds, convergence of viscosity approximations in the literature has been established in the context of elastodynamics by Shearer [25] and Shearer and Serre [26]. The objective in that context is to establish the reduction of the generalized Young measure to a point mass and to show strong convergence.

Theorem 4.1.2 (Weakly dissipative source). *Let $\{u_\varepsilon\}$ be a family of smooth solutions of* (3.3) *on $\mathbb{R}\times[0,T]$ emanating from smooth initial data. The family $\{u_\varepsilon\}$ is assumed to decay fast at infinity. Let the hypotheses of Proposition 3.1.2 remain valid so that the stability estimate* (3.9) *holds true with $\eta-q$ entropy-entropy flux pair satisfying* (H1)*, and A a symmetric, positive-definite matrix subject to* (H2)*. Then, for entropy pairs $(\bar\eta,\bar q)$ satisfying*

$$\|\bar\eta\|_{L^\infty}, \|\bar q\|_{L^\infty}, \|\mathrm{D}\bar\eta\|_{L^\infty}, \|\mathrm{D}^2\bar\eta\|_{L^\infty} \le C \tag{4.1}$$

and

$$|\mathrm{D}\bar\eta(v)G(v)| \le C\big(M-\mathrm{D}\eta(v)G(v)\big), \quad \forall v\in\mathbb{R}^n\,, \tag{4.2}$$

the family

$$\big\{\partial_t\bar\eta(u^\varepsilon)+\partial_x\bar q(u^\varepsilon)\big\}_\varepsilon \textit{ lies in a compact set of } H^{-1}_{loc}(\mathbb{R}\times[0,T]). \tag{4.3}$$

Proof. Let $\{u^\varepsilon\}$ be a family of smooth solutions of (3.3) on $\mathbb{R}\times[0,T]$. The goal is to control the dissipation measure and to establish (4.3) for a class of entropy-entropy flux pairs (η,q). It suffices to establish (4.3) for entropy-entropy flux pairs $(\bar\eta,\bar q)$ satisfying (4.1). This class of entropy pairs has been used in the literature in a different context in order to establish the reduction of the generalized Young measure to a point mass and to show strong convergence.

Starting from (3.3) we obtain

$$\begin{aligned}\partial_t\bar\eta(u^\varepsilon)+\partial_x\bar q(u^\varepsilon) &= \varepsilon\partial_x\left(\mathrm{D}\bar\eta(u^\varepsilon)Au^\varepsilon_x\right)-\varepsilon\partial_t\left(\mathrm{D}\bar\eta(u^\varepsilon)u^\varepsilon_t\right)\\ &\quad -\varepsilon u_x^{\varepsilon\top}\mathrm{D}^2\bar\eta(u^\varepsilon)Au^\varepsilon_x+\varepsilon u_t^{\varepsilon\top}\mathrm{D}^2\bar\eta(u^\varepsilon)u^\varepsilon_t+\mathrm{D}\bar\eta(u^\varepsilon)G(u^\varepsilon)\\ &:= I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

From (3.9) and (4.1), the terms I_1, I_2 lie in compact set of H^{-1}, and the terms I_3, I_4 are bounded in L^1. By (3.9) $\mathrm{D}\eta G(u^\varepsilon)$ is bounded in L_1 by initial data (due to the weak dissipation assumption) and therefore by (4.2) the term $\mathrm{D}\bar\eta G(u^\varepsilon)$ is in a bounded set of L_1 as well. Therefore, by Murat's lemma [22], the sum $\sum I_i$ lies in a bounded set of $W^{-1,\infty}$. □

Following, similar line of argument an analogous result for systems of hyperbolic balance laws with general source term is established.

Theorem 4.1.3 (General source). *Let $\{u_\varepsilon\}$ be a family of smooth solutions of* (3.3) *on $\mathbb{R}\times[0,T]$ emanating from smooth initial data. The family $\{u_\varepsilon\}$ is assumed to decay fast at infinity. Let the hypothesis of Proposition 3.1.3 remain valid so that the stability estimate* (3.13) *holds true with $\eta-q$ entropy-entropy flux pair satisfying* (H1)*, and A a symmetric, positive-definite matrix subject to* (H2)*. Then, for entropy pairs $(\bar\eta,\bar q)$ satisfying*

$$\|\bar\eta\|_{L^\infty}, \|\bar q\|_{L^\infty}, \|\mathrm{D}\bar\eta\|_{L^\infty}, \|\mathrm{D}^2\bar\eta\|_{L^\infty} \le C \tag{4.4}$$

the family

$$\Big\{\partial_t\bar\eta(u_\varepsilon)+\partial_x\bar q(u_\varepsilon)\Big\}_\varepsilon \textit{ lies in a compact set of } H^{-1}_{loc}(\mathbb{R}\times[0,T]). \tag{4.5}$$

Proof. The proof follows similar line of argument as the one in Theorem 4.1.2. Starting from (3.3) we obtain

$$\begin{aligned}\partial_t\bar{\eta}(u^\varepsilon)+\partial_x\bar{q}(u^\varepsilon) &= \varepsilon\partial_x\left(\mathrm{D}\bar{\eta}(u^\varepsilon)Au^\varepsilon_x\right)-\varepsilon\partial_t\left(\mathrm{D}\bar{\eta}(u^\varepsilon)u^\varepsilon_t\right)\\ &\qquad -\varepsilon {u^\varepsilon_x}^\top \mathrm{D}^2\bar{\eta}(u^\varepsilon)Au^\varepsilon_x+\varepsilon {u^\varepsilon_t}^\top\mathrm{D}^2\bar{\eta}(u^\varepsilon)u^\varepsilon_t+\mathrm{D}\bar{\eta}(u^\varepsilon)G(u^\varepsilon)\\ &:= I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

From (3.13) and (4.4), the terms I_1, I_2 lie in compact set of H^{-1}, the terms I_3, I_4 are bounded in L^1, whereas

$$|I_5| = |\mathrm{D}\bar{\eta}(u)^\top G(u)| \leq c|u|^2$$

is by Lemma 3.1.3 bounded in L^1. Therefore, by Murat's lemma [22], the sum $\sum I_i$ lies in a bounded set of $W^{-1,\infty}$. □

5 Error estimates via the relative entropy method, $d = 1$

In this section, we establish the convergence of solutions of (3.3) to solutions of (3.1) before the formation of shocks. In the spirit of [1] we use the modified relative entropy method [12] by introducing a functional $H^{rel}(\bar{u}, u^\varepsilon)$, which monitors the difference between the solutions $\bar{u}$ to the equilibrium and the solutions u^ε to the relaxation systems. The presence of the source G in our work, however, requires us to modify the method significantly. More specifically, in order to treat the weakly dissipative source G, which satifies typically no growth restrictions, we need to introduce the relative potential $R^{rel}(\bar{u}, u^\varepsilon)$ (see (5.18) in Section 6.3). This potential becomes part of a Lyapunov functional monitoring the evolution of the difference between the two sources. In the case of the general source G the terms associated with the source are treated as error (cf. Section 6.4).

5.1 The decay functional and relative entropy identity.

Let $\eta - q$ be an entropy-entropy flux pair satisfying (H1). We define the corresponding relative entropy-entropy flux pair by

$$\begin{aligned}H^{rel}(\bar{u},u^\varepsilon) &= \eta\big(u^\varepsilon+\varepsilon(u^\varepsilon-\bar{u})_t\big)-\eta(\bar{u})-\mathrm{D}\eta(\bar{u})\big(u^\varepsilon+\varepsilon(u^\varepsilon-\bar{u})_t-\bar{u}\big)\\ Q^{rel}(\bar{u},u^\varepsilon) &= q(u^\varepsilon)-q(\bar{u})-\mathrm{D}\eta(\bar{u})(F(u^\varepsilon)-F(\bar{u}))\end{aligned} \tag{5.1}$$

and set the functional

$$\begin{aligned}\mathcal{G}(\bar{u},u^\varepsilon) &= H_R(\bar{u},u^\varepsilon)\\ &\qquad+\varepsilon^2(u^\varepsilon-\bar{u})_t^\top\big(\alpha I-\overline{D^2\eta}\big)(u^\varepsilon-\bar{u})_t\\ &\qquad+\varepsilon^2\alpha(u^\varepsilon-\bar{u})_x^\top A(u^\varepsilon-\bar{u})_x\,,\end{aligned} \tag{5.2}$$

where A is a symmetric, positive definite matrix, $\alpha > 0$ a fixed constant defined in (H1) and

$$\overline{D^2\eta} = \int_0^1\int_0^s D^2\eta\big(u^\varepsilon+\varepsilon\tau(u^\varepsilon-\bar{u})_t\big)\,d\tau ds\,. \tag{5.3}$$

Lemma 5.1.1 (Relative entropy identity). *Let $\bar u$, u^ε be smooth solutions to* (3.1), (3.3), *respectively. Then, the following energy identity holds*

$$
\begin{aligned}
&\partial_t\, \mathcal{G}(\bar u, u^\varepsilon) + \partial_x Q^{rel}(\bar u, u^\varepsilon) + \varepsilon\alpha\big|(u^\varepsilon-\bar u)_t + \mathrm{D}F(u^\varepsilon)(u^\varepsilon-\bar u)_x\big|^2\\
&\quad + \varepsilon\Big\{(u^\varepsilon-\bar u)_t^\top\Big(\alpha I - \mathrm{D}^2\eta(u^\varepsilon)\Big)(u^\varepsilon-\bar u)_t\Big\}\\
&\quad + \varepsilon\Big\{(u^\varepsilon-\bar u)_x^\top\Big(\mathrm{D}^2\eta(u^\varepsilon)A - \alpha\,\mathrm{D}F(u^\varepsilon)^\top\mathrm{D}F(u^\varepsilon)\Big)(u^\varepsilon-\bar u)_x\Big\}\\
&= \partial_x\Big\{\varepsilon\big(\mathrm{D}\eta(u^\varepsilon)-\mathrm{D}\eta(\bar u)\big)A(u^\varepsilon-\bar u)_x + 2\alpha\varepsilon^2\big(A(u^\varepsilon-\bar u)_x\big)^\top(u^\varepsilon-\bar u)_t\Big\}\\
&\quad - \big(\mathrm{D}^2\eta(\bar u)\bar u_x\big)^\top\Big(F(u^\varepsilon)-F(\bar u)-\mathrm{D}F(\bar u)(u^\varepsilon-\bar u)\Big)\\
&\quad + \big(a_1+a_2+b_1+b_2+2\varepsilon\alpha(c_1+c_2)\big) + \big(d_1+d_2+2\varepsilon\alpha d_3\big)\,,
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
a_1 &= \varepsilon\Big(\big(\mathrm{D}^2\eta(u^\varepsilon)-\mathrm{D}^2\eta(\bar u)\big)\bar u_t\Big)^\top(u^\varepsilon-\bar u)_t\\
a_2 &= -\varepsilon\big(\mathrm{D}\eta(u^\varepsilon)-\mathrm{D}\eta(\bar u)\big)\bar u_{tt}\\
b_1 &= \varepsilon\Big(\big(\mathrm{D}^2\eta(u^\varepsilon)-\mathrm{D}^2\eta(\bar u)\big)\bar u_x\Big)^\top A(u^\varepsilon-\bar u)_x\\
b_2 &= -\varepsilon\big(\mathrm{D}\eta(u^\varepsilon)-\mathrm{D}\eta(\bar u)\big)A\bar u_{xx}\\
c_1 &= \varepsilon(A\bar u_{xx}-\bar u_{tt})^\top(u^\varepsilon-\bar u)_t\\
c_2 &= -\Big(\big(\mathrm{D}F(u^\varepsilon)-\mathrm{D}F(\bar u)\big)\bar u_x\Big)^\top(u^\varepsilon-\bar u)_t
\end{aligned}
\tag{5.5}
$$

are the error terms, and

$$
\begin{aligned}
d_1 &= \big(\mathrm{D}\eta(u^\varepsilon)-\mathrm{D}\eta(\bar u)\big)(G(u^\varepsilon)-G(\bar u))\\
d_2 &= G(\bar u)^\top\Big(\mathrm{D}\eta(u^\varepsilon)-\mathrm{D}\eta(\bar u)-\mathrm{D}^2\eta(\bar u)(u^\varepsilon-\bar u)\Big)\\
d_3 &= \big(G(u^\varepsilon)-G(\bar u)\big)^\top(u^\varepsilon-\bar u)_t\,.
\end{aligned}
\tag{5.6}
$$

are the terms associated with the source G.

Proof. By (H1), (3.1), and (3.3) we have

$$
\begin{aligned}
&\partial_t\big(\eta(u^\varepsilon)-\eta(\bar u)\big) + \partial_x\big(q(u^\varepsilon)-q(\bar u)\big)\\
&\quad = \varepsilon\big(D\eta(u^\varepsilon)Au^\varepsilon_{xx} - D\eta(u^\varepsilon)u^\varepsilon_{tt}\big) + D\eta(u^\varepsilon)G(u^\varepsilon) - D\eta(\bar u)G(\bar u)\,.
\end{aligned}
\tag{5.7}
$$

Similarly,

$$
\partial_t(u^\varepsilon-\bar u) + \partial_x(F(u^\varepsilon)-F(\bar u)) = \varepsilon(Au^\varepsilon_{xx}-u^\varepsilon_{tt}) + G(u^\varepsilon)-G(\bar u)
$$

and hence, after multiplying the above identity by $\mathrm{D}\eta(\bar u)$, we have

$$
\begin{aligned}
&\partial_t\big(\mathrm{D}\eta(\bar u)(u^\varepsilon-\bar u)\big) + \partial_x\big(\mathrm{D}\eta(\bar u)\big(F(u^\varepsilon)-F(\bar u)\big)\big)\\
&\quad = \big(\mathrm{D}^2\eta(\bar u)\bar u_t\big)^\top(u^\varepsilon-\bar u) + \partial_x\big(\mathrm{D}\eta(\bar u)\big)(F(u^\varepsilon)-F(\bar u))\\
&\qquad + \varepsilon\big(\mathrm{D}\eta(\bar u)Au^\varepsilon_{xx} - \mathrm{D}\eta(\bar u)u^\varepsilon_{tt}\big) + \mathrm{D}\eta(\bar u)(G(u^\varepsilon)-G(\bar u))\,.
\end{aligned}
\tag{5.8}
$$

The existence of the entropy-pair $\eta - q$ is equivalent to the property

$$
\mathrm{D}^2\eta(v)\mathrm{D}F(v) = \mathrm{D}F(v)^\top\mathrm{D}^2\eta(v), \quad \forall v\in\mathbb{R}^n
$$

and hence, using (3.1), the first term on the right-hand side of (5.8) can be expressed as

$$\big(\mathrm{D}^2\eta(\bar u)\bar u_t\big)^\top (u^\varepsilon - \bar u) = -\big(\mathrm{D}^2\eta(\bar u)\bar u_x\big)^\top \mathrm{D}F(\bar u)(u^\varepsilon - \bar u) + G(\bar u)^\top \mathrm{D}^2\eta(\bar u)(u^\varepsilon - \bar u)\,.$$

Combining (5.7), (5.8) and the above identity we obtain

$$\begin{aligned}
&\partial_t\big(\eta(u^\varepsilon) - \eta(\bar u) - D\eta(\bar u)(u^\varepsilon - \bar u)\big)\\
&\quad + \partial_x\big(q(u^\varepsilon) - q(\bar u) - \mathrm{D}\eta(\bar u)(F(u^\varepsilon) - F(\bar u))\big)\\
&\qquad = -\big(\mathrm{D}^2\eta(\bar u)\bar u_x\big)^\top \big(F(u^\varepsilon) - F(\bar u) - \mathrm{D}F(\bar u)(u^\varepsilon - \bar u)\big)\\
&\qquad\quad + \varepsilon\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)Au^\varepsilon_{xx} - \varepsilon\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)u^\varepsilon_{tt}\\
&\qquad\quad + \big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)\big(G(u^\varepsilon) - G(\bar u)\big)\\
&\qquad\quad + G(\bar u)^\top\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u) - \mathrm{D}^2\eta(\bar u)(u^\varepsilon - \bar u)\big).
\end{aligned} \tag{5.9}$$

Next, we express the second and third terms on the right-hand side of (5.9) as

$$\begin{aligned}
&\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)u^\varepsilon_{tt}\\
&\quad = \partial_t\Big(\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)(u^\varepsilon - \bar u)_t\Big) - \big(\mathrm{D}^2\eta(u^\varepsilon)(u^\varepsilon - \bar u)_t\big)^\top (u^\varepsilon - \bar u)_t\\
&\qquad - \Big(\big(\mathrm{D}^2\eta(u^\varepsilon) - \mathrm{D}^2\eta(\bar u)\big)\bar u_t\Big)^\top (u^\varepsilon - \bar u)_t + \big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)\bar u_{tt}
\end{aligned}$$

$$\begin{aligned}
&\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)Au^\varepsilon_{xx}\\
&\quad = \partial_x\Big(\big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)A(u^\varepsilon - \bar u)_x\Big) - \big(\mathrm{D}^2\eta(u^\varepsilon)(u^\varepsilon - \bar u)_x\big)^\top A(u^\varepsilon - \bar u)_x\\
&\qquad - \Big(\big(\mathrm{D}^2\eta(u^\varepsilon) - \mathrm{D}^2\eta(\bar u)\big)\bar u_x\Big)^\top A(u^\varepsilon - \bar u)_x + \big(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u)\big)^\top A\bar u_{xx}
\end{aligned}$$

and observe that

$$\eta\big(u^\varepsilon + \varepsilon(u^\varepsilon - \bar u)_t\big) = \eta(u^\varepsilon) + \varepsilon\mathrm{D}\eta(u^\varepsilon)(u^\varepsilon - \bar u)_t + \varepsilon^2(u^\varepsilon - \bar u)_t^\top\,\overline{\mathrm{D}^2\eta}\,(u^\varepsilon - \bar u)_t\,.$$

Then (5.1), (5.9) and the last three identities imply

$$\begin{aligned}
&\partial_t\Big\{H^{rel}(\bar u, u^\varepsilon) - \varepsilon^2(u^\varepsilon - \bar u)_t^\top\,\overline{\mathrm{D}^2\eta}\,(u^\varepsilon - \bar u)_t\Big\} + \partial_x Q^{rel}(\bar u, u^\varepsilon)\\
&\quad + \varepsilon\Big\{\big(\mathrm{D}^2\eta(u^\varepsilon)(u^\varepsilon - \bar u)_x\big)^\top (u^\varepsilon - \bar u)_x - \big(\mathrm{D}^2\eta(u^\varepsilon)(u^\varepsilon - \bar u)_t\big)^\top (u^\varepsilon - \bar u)_t\Big\}\\
&= \partial_x\Big\{\varepsilon(\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u))A(u^\varepsilon - \bar u)_x\Big\}\\
&\quad - \big(\mathrm{D}^2\eta(\bar u)\bar u_x\big)^\top\Big(F(u^\varepsilon) - F(\bar u) - \mathrm{D}F(\bar u)(u^\varepsilon - \bar u)\Big)\\
&\quad + a_{1t} + a_{2t} + b_{1x} + b_{2x} + d_1 + d_2
\end{aligned} \tag{5.10}$$

with the last six terms on the right-hand side of the above identity defined in $(5.5)_{1,2,3,4}$ and $(5.6)_{1,2}$.

The identity (5.10) is supplemented by a correction accounting for the fact that the third term is indefinite. The correcting identity is obtained by multiplying the equation

$$\begin{aligned}
(u^\varepsilon - \bar u)_t + \mathrm{D}F(u^\varepsilon)(u^\varepsilon - \bar u)_x &= \varepsilon A(u^\varepsilon - \bar u)_{xx} - \varepsilon(u^\varepsilon - \bar u)_{tt} + \varepsilon(A\bar u_{xx} - \bar u_{tt})\\
&\quad + (G(u^\varepsilon) - G(\bar u)) - (\mathrm{D}F(u^\varepsilon) - \mathrm{D}F(\bar u))\bar u_x
\end{aligned}$$

by $(u^\varepsilon - \bar u)_t$ and integrating by parts, which leads to

$$\begin{aligned}\partial_t\Big\{\tfrac12\,\varepsilon|u^\varepsilon_t-\bar u_t|^2+\tfrac12\,\varepsilon(u^\varepsilon-\bar u)_x^\top A(u^\varepsilon-\bar u)_x\Big\}&\\ +|u^\varepsilon_t-\bar u_t|^2+\big(\mathrm{D}F(u^\varepsilon)(u^\varepsilon-\bar u)_x\big)^\top(u^\varepsilon-\bar u)_t&\\ =\partial_x\Big\{\varepsilon\big(A(u^\varepsilon-\bar u)_x\big)^\top(u^\varepsilon-\bar u)_t\Big\}+c_{1t}+c_{2t}+d_3&\end{aligned}\tag{5.11}$$

with the terms on the right-hand side defined in $(5.5)_{5,6}$ and $(5.6)_3$.

Finally, multiplying (5.11) by $2\alpha\varepsilon$ and adding the resulting identity to (5.10) we obtain (5.4). □

5.2 Preliminary estimate of $\mathcal{G}$

In this section we establish a preliminary estimate for the functional $\mathcal{G}(\bar u, u^\varepsilon)$ employed in the proofs of Theorem 5.3.1 and Theorem 5.4.1. For this purpose we define

$$\Psi(t) := \int_{\mathbb{R}} |\bar u - u^\varepsilon|^2 + \varepsilon^2|(\bar u - u^\varepsilon)_x|^2 + \varepsilon^2|(\bar u - u^\varepsilon)_t|^2\,dx \tag{5.12}$$

used in our further analysis, where $\bar u$, u are smooth solutions to the equilibrium and relaxation system, respectively.

Lemma 5.2.1. *Let $\bar u$, u^ε be smooth solutions of* (3.1), (3.2), *respectively and suppose that both $\bar u$, u^ε decay sufficiently fast at infinity. Suppose that:*

(a1) (H1) *holds true and the positive definite, symmetric matrix A, is such that the subcharacteristic condition* (H2) *is valid.*

(a2) For some $M>0$

$$|D^2F(u)| \le M, \quad |D^3\eta(u)| \le M, \quad u\in\mathbb{R}^n.$$

Then,

(i) There exists $c_1, c_2 > 0$ independent of $\varepsilon>0$ such that

$$c_1\Psi(t) \le \int_{\mathbb{R}} \mathcal{G}(\bar u, u^\varepsilon)\,dx \le c_2\Psi(t)\,. \tag{5.13}$$

(ii) The functional $\mathcal{G}(\bar u, u^\varepsilon)$ is positive definite and satisfies

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \mathcal{G}(\bar u, u^\varepsilon)\,dx + \varepsilon\bar c\int_{\mathbb{R}} |u^\varepsilon_x - \bar u_x|^2 + |u^\varepsilon_t - \bar u_t|^2\,dx&\\ \le C(T,\bar u)\Big(\Psi(t)+\varepsilon^2\Big) + \int_{\mathbb{R}}\Big(d_1+d_2+2\varepsilon\alpha d_3\Big)dx&\end{aligned}\tag{5.14}$$

with d_1, d_2, d_3 defined in (5.6) *and $C = C(T,\bar u) > 0$ independent of ε.*

Proof. From (H1) and $(5.1)_1$ we have

$$\beta|u^\varepsilon + \varepsilon(u^\varepsilon - \bar{u})_t - \bar{u}|^2 \le H^{rel}(\bar{u}, u^\varepsilon) \le \alpha|u^\varepsilon + \varepsilon(u^\varepsilon - \bar{u})_t - \bar{u}|^2 . \tag{5.15}$$

Also, (H1) and (5.3) imply

$$\alpha\mathbf{I} \ge \alpha\mathbf{I} - \overline{D^2\eta} = \alpha I - \int_0^1 \int_0^s D^2\eta\big(u^\varepsilon + \varepsilon\tau(u^\varepsilon - \bar{u})_t\big)\, d\tau ds \ge \tfrac{1}{2}\alpha I. \tag{5.16}$$

Combining (5.2), (5.15), (5.16) and recalling that A is symmetric, positive definite we get (5.13).

Next, integrating (5.4) we use (H2) and (5.16) to conclude

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \mathcal{G}(\bar{u}, u^\varepsilon)\, dx + \varepsilon\bar{c}\int_{\mathbb{R}} |(\bar{u} - u^\varepsilon)_x|^2 + |(\bar{u} - u^\varepsilon)_t|^2\, dx \\ &\quad \le \int_{\mathbb{R}} \Big\{ \big|\big(\mathrm{D}^2\eta(\bar{u})\bar{u}_x\big)^\top \big(F(u^\varepsilon) - F(\bar{u}) - \mathrm{D}F(\bar{u})(u^\varepsilon - \bar{u})\big)\big| \\ &\qquad + \big|a_1 + a_2 + b_1 + b_2 + 2\varepsilon\alpha(c_1 + c_2)\big| + d_1 + d_2 + 2\varepsilon\alpha d_3 \Big\}\, dx \end{aligned} \tag{5.17}$$

By $(a3)$ we have

$$\int_{\mathbb{R}} \big|\big(\mathrm{D}^2\eta(\bar{u})\bar{u}_x\big)^\top \big(F(u^\varepsilon) - F(\bar{u}) - \mathrm{D}F(\bar{u})(u^\varepsilon - \bar{u})\big)\big|\, dx \le C\,\|u^\varepsilon - \bar{u}\|_{L^2}^2$$

and the error terms in (5.5) are estimated by

$$\begin{aligned} &\|a_1\|_{L^1} \le \varepsilon C\,\|\bar{u}_t\|_{L^\infty}\,\|u^\varepsilon - \bar{u}\|_{L^2}\,\|u_t^\varepsilon - \bar{u}_t\|_{L^2} \qquad \|a_2\|_{L^1} \le \varepsilon C\,\|\bar{u}_{tt}\|_{L^2}\,\|u^\varepsilon - \bar{u}\|_{L^2} \\ &\|b_1\|_{L^1} \le \varepsilon C\,\|\bar{u}_x\|_{L^\infty}\,\|u^\varepsilon - \bar{u}\|_{L^2}\,\|u_x^\varepsilon - \bar{u}_x\|_{L^2} \qquad \|b_2\|_{L^1} \le \varepsilon C\,\|\bar{u}_{xx}\|_{L^2}\,\|u^\varepsilon - \bar{u}\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} &\|\varepsilon c_1\|_{L^1} \le \varepsilon^2 C\big(\|\bar{u}_{tt}\|_{L^2} + \|\bar{u}_{xx}\|_{L^2}\big)\,\|u_t^\varepsilon - \bar{u}_t\|_{L^2} \\ &\|\varepsilon c_2\|_{L^1} \le \varepsilon C\,\|\bar{u}_x\|_{L^\infty}\,\|u^\varepsilon - \bar{u}\|_{L^2}\,\|u_t^\varepsilon - \bar{u}_t\|_{L^2} , \end{aligned}$$

where C is a generic constant that depends on α, M and norms of $\bar{u}$. Then, by (5.13), (5.17) and the above estimates we obtain (5.14). □

5.3 Error estimates. Weakly dissipative source $G(u)$

To establish the convergence result for weakly dissipative source we introduce the *relative potential*

$$R^{rel}(\bar{u}, u^\varepsilon) := R(u^\varepsilon) - R(\bar{u}) - \mathrm{D}R(\bar{u})(u - \bar{u}) \ge 0 \tag{5.18}$$

which is well-defined whenever $G \in C^1(\mathbb{R}^n)$. As we will see in the next theorem the smoothness of G, which in our work is in general assumed to be $C(\mathbb{R}^n)$, will have an impact on the rate of convergence.

Theorem 5.3.1. *Let $\bar{u}(x,t)$ be a smooth solution of the equilibrium system* (3.1)*, defined on $\mathbb{R}^d \times [0,T]$. Let $\{u^\varepsilon\}$ be a family of smooth solutions of the relaxation system* (3.2) *on $\mathbb{R} \times [0,T]$. Suppose that both $\bar{u}$ and u^ε decay sufficiently fast at infinity and that:*

(a1) (H1) holds true and the positive definite, symmetric matrix A, is such that the subcharacteristic condition (H2) *is valid.*

(a2) For some $M > 0$

$$|D^2F(u)| \le M, \quad |D^3\eta(u)| \le M, \quad u \in \mathbb{R}^n.$$

(a3) The conditions (H3-a), (H4) *on the source term hold true.*

Then for all $t \in [0,T]$

$$\Psi(t) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,t))\,dx \;\le\; C\Big(\Psi(0) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,0))\,dx + \varepsilon\Big) \tag{5.19}$$

with $C = C(\bar u, \alpha, \beta, \nu, M, T) > 0$ independent of ε.

If, in addition, $G \in C^2(\mathbb{R}^n)$ then for all $t \in [0,T]$

$$\Psi(t) + \varepsilon \int_{\mathbb{R}} \big[R^{rel}(\bar u, u^\varepsilon)\big](x,t)\,dx \;\le\; C\Big(\Psi(0) + \varepsilon \int_{\mathbb{R}} \big[R^{rel}(\bar u, u^\varepsilon)\big](x,0)\,dx + \varepsilon^2\Big)\,. \tag{5.20}$$

Proof. By (H1), (H3-b), and (5.6) we obtain

$$\begin{aligned} d_1 &\le (\mathrm{D}\eta(u^\varepsilon) - \mathrm{D}\eta(\bar u))(G(u^\varepsilon) - G(\bar u)) \le 0 \\ d_2 &\le \alpha \,\|G(\bar u)\|_{L^\infty} |u^\varepsilon - \bar u|^2 \\ \varepsilon d_3 &\le -\varepsilon\,\partial_t R(u^\varepsilon) + \varepsilon C_R (1 + R(u^\varepsilon))|\bar u_t| + \varepsilon |G(\bar u)||u^\varepsilon_t - \bar u_t|. \end{aligned} \tag{5.21}$$

Then, combining (5.14) and (5.21) we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \mathcal{G}(\bar u, u^\varepsilon)\,dx + \varepsilon \bar c \int_{\mathbb{R}} |u^\varepsilon_x - \bar u_x|^2 + |u^\varepsilon_t - \bar u_t|^2\,dx \\ &\quad\le C\Big(\Psi(t) + \varepsilon^2\Big) + 2\varepsilon\alpha C_R \left(\|\bar u_t\|_{L^1} + \|\bar u_t\|_{L^\infty} \int_{\mathbb{R}} R(u^\varepsilon)\,dx\right) \\ &\qquad - 2\varepsilon\alpha \frac{d}{dt}\int_{\mathbb{R}} R(u^\varepsilon)\,dx + \frac{2\varepsilon\alpha^2}{\bar c}\int_{\mathbb{R}} |G(\bar u)|^2\,dx + \frac{\varepsilon \bar c}{2}\int_{\mathbb{R}} |u^\varepsilon_t - \bar u_t|^2\,dx. \end{aligned} \tag{5.22}$$

Integrating the above inequality, and using (H1), (H4), (5.13) and (5.12), we obtain

$$\begin{aligned} &\Psi(t) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,t))\,dx \\ &\quad \le C\bigg(\Psi(0) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,0))\,dx + \int_0^t \Big\{\Psi(s) + \varepsilon \int_{\mathbb{R}} R(u^\varepsilon(x,s))\,dx\Big\}\,ds + \varepsilon t + \varepsilon^2 t\bigg) \end{aligned}$$

and conclude (5.19) via the Gronwall lemma.

Suppose now that $G \in C^2(\mathbb{R}^n)$. Then, using (H4) and (5.18), we obtain

$$\begin{aligned} \partial_t\big(R^{rel}(\bar u, u^\varepsilon)\big) &= -G(u^\varepsilon)^\top u^\varepsilon_t + G(\bar u)^\top \bar u_t + G(\bar u)^\top (u^\varepsilon - \bar u)_t \\ &\quad - (\mathrm{D}^2 R(\bar u)\bar u_t)^\top (u^\varepsilon - \bar u) \\ &= (G(\bar u) - G(u^\varepsilon))^\top u^\varepsilon_t - (\mathrm{D}^2 R(\bar u)\bar u_t)^\top (u^\varepsilon - \bar u)\,. \end{aligned} \tag{5.23}$$

Then, using (H4), $(5.6)_3$ and (5.23), we obtain

$$\begin{aligned} d_3 &= \big(G(u^\varepsilon) - G(\bar u)\big)^\top (u^\varepsilon - \bar u)_t \\ &= -\partial_t\big(R^{rel}(\bar u, u^\varepsilon)\big) - (\mathrm{D}^2 R(\bar u)\bar u_t)^\top (u^\varepsilon - \bar u) - (G(u^\varepsilon) - G(\bar u))^\top \bar u_t \\ &= -\partial_t\big(R^{rel}(\bar u, u^\varepsilon)\big) + \bar u_t^\top \big(\mathrm{D}R(u^\varepsilon) - \mathrm{D}R(\bar u) - \mathrm{D}^2 R(\bar u)(u^\varepsilon - \bar u)\big)\,. \end{aligned} \tag{5.24}$$

Then, combining (5.14) with $(5.21)_{1,2}$ and (5.24), we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb R} \mathcal G(\bar u, u^\varepsilon)\,dx + \varepsilon\bar c \int_{\mathbb R} |u^\varepsilon_x - \bar u_x|^2 + |u^\varepsilon_t - \bar u_t|^2\,dx \\ &\quad \le\; C(T,\bar u)\Big(\Psi(t) + \varepsilon^2\Big) - 2\varepsilon\alpha \frac{d}{dt}\int_{\mathbb R} R^{rel}(\bar u, u^\varepsilon)\,dx \\ &\qquad + 2\varepsilon\alpha \int_{\mathbb R} \Big\{ \bar u_t^\top \big(\mathrm{D}R(u^\varepsilon) - \mathrm{D}R(\bar u) - \mathrm{D}^2 R(\bar u)(u^\varepsilon - \bar u)\big)\Big\}\,dx\,. \end{aligned} \tag{5.25}$$

Now, we estimate the last term on the right-hand side of (5.25). By assumption on $\bar u$ there exists a compact set $\mathcal K \subset \mathbb R^n$ such that

$$\bar u(x,t) \in \mathcal K\,, \quad (x,t) \in \mathbb R \times [0,T].$$

Thus, we can take large enough $M > 0$ such that

$$\mathcal K \subset B_M \quad \text{and} \quad |\bar u(x,t) - v| > 1 \quad \text{for all} \quad v \in {B_M}^C\,,\ (x,t) \in \mathbb R \times [0,T], \tag{5.26}$$

where B_M denotes a ball of radius M. Then, since $G = -\mathrm{D}R \in C^2(\mathbb R^n)$, we obtain

$$\lambda_G^M := \sup_{v\in B_M} \Big(|R(v)| + |\mathrm{D}R(v)| + |\mathrm{D}^2 R(v)| + |\mathrm{D}^3 R(v)|\Big) < \infty\,. \tag{5.27}$$

Fix $(x,t) \in \mathbb R \times [0,T]$. Suppose $u(x,t) \in \mathcal B_M$. Then by (5.27)

$$|\mathrm{D}R(u^\varepsilon) - \mathrm{D}R(\bar u) - \mathrm{D}^2 R(\bar u)(u^\varepsilon - \bar u)| \le \lambda_G^M |u^\varepsilon - \bar u|^2\,. \tag{5.28}$$

Suppose now that $u^\varepsilon(x,t) \in {\mathcal B_M}^C$. Then, by (5.26) we have $|u^\varepsilon(x,t) - \bar u(x,t)| > 1$. Hence

$$\begin{aligned} 0 \le R(u^\varepsilon(x,t)) &= R^{rel}(\bar u, u^\varepsilon) + R(\bar u) + \mathrm{D}R(\bar u)(u - \bar u) \\ &\le \max(R^{rel}, |u^\varepsilon - \bar u|^2)\Big(1 + 2\lambda_G^M\Big)\,. \end{aligned} \tag{5.29}$$

Then, using (5.27), (5.29) and (H4), we obtain

$$\begin{aligned} &|\mathrm{D}R(u^\varepsilon) - \mathrm{D}R(\bar u) - \mathrm{D}^2 R(\bar u)(u^\varepsilon - \bar u)| \\ &\qquad \le\; 2\max(R(u^\varepsilon), |u^\varepsilon - \bar u|^2)\bigg(\frac{|\mathrm{D}R(u^\varepsilon)|}{R(u^\varepsilon)+1} + \lambda_G^M\bigg) \\ &\qquad \le\; 2\max(R(u^\varepsilon), |u^\varepsilon - \bar u|^2)\big(C_R + \lambda_G^M\big) \\ &\qquad \le C\max(R^{rel}, |u^\varepsilon - \bar u|^2) \end{aligned} \tag{5.30}$$

for some $C > 0$ that depends on λ_G^M and C_R.

Since (x, t) is arbitrarily chosen, combining (5.28) and (5.30), we conclude

$$|\mathrm{D}R(u^\varepsilon) - \mathrm{D}R(\bar{u}) - \mathrm{D}^2R(\bar{u})(u^\varepsilon - \bar{u})| \le C \max(R^{rel}(u^\varepsilon), |u^\varepsilon - \bar{u}|^2) \tag{5.31}$$

for all $(x, t) \in \mathbb{R} \times [0, T]$. Thus

$$\begin{aligned} 2\varepsilon\alpha \int_{\mathbb{R}} &\Big\{ \bar{u}_t^\top \big(\mathrm{D}R(u^\varepsilon) - \mathrm{D}R(\bar{u}) - \mathrm{D}^2R(\bar{u})(u^\varepsilon - \bar{u})\big) \Big\}\, dx \\ &\le C\bigg(\varepsilon \int_{\mathbb{R}} R^{rel}(\bar{u}, u^\varepsilon)\, dx + \varepsilon \int_{\mathbb{R}} |\bar{u} - u^\varepsilon|^2\, dx \bigg). \end{aligned} \tag{5.32}$$

Then, integrating the inequality (5.25) in time, and using the (5.13), (5.12), and (5.32), we obtain

$$\begin{aligned} &\Psi(t) + \varepsilon \int_{\mathbb{R}} \big[R^{rel}(\bar{u}, u^\varepsilon)\big](x, t)\, dx \\ &\le C\bigg(\Psi(0) + \varepsilon \int_{\mathbb{R}} \big[R^{rel}(\bar{u}, u^\varepsilon)\big](x, 0)\, dx \\ &\qquad + \int_0^\tau \Big\{ \Psi(\tau) + \varepsilon \int_{\mathbb{R}} \big[R^{rel}(\bar{u}, u^\varepsilon)\big](x, \tau)\, dx \Big\}\, d\tau + \varepsilon^2 \bigg) \end{aligned} \tag{5.33}$$

which along with the Gronwall lemma implies (5.19). ☐

5.4 Error estimates. General source $G(u)$

We now consider the source term that satisfies the alternative hypothesis (H3-b).

Theorem 5.4.1. *Let $\bar{u}(x, t)$ be a smooth solution of the equilibrium system* (3.1)*, defined on $\mathbb{R} \times [0, T]$. Let $\{u^\varepsilon\}$ be a family of smooth solutions of the relaxation system* (3.2) *on $\mathbb{R} \times [0, T]$. Suppose that both $\bar{u}$ and u^ε decay sufficiently fast at infinity and that:*

(a1) (H1) holds true and the positive definite, symmetric matrix A, is such that the subcharacteristic condition (H2) *is valid.*

(a2) For some $M > 0$

$$|\mathrm{D}^2F(u)| \le M, \quad |\mathrm{D}^3\eta(u)| \le M, \quad u \in \mathbb{R}^n.$$

(a3) The condition (H3-b) on the source term holds true.

Then,

$$\Psi(t) \le C\big(\Psi(0) + \varepsilon^2\big), \quad t \in [0, T] \tag{5.34}$$

with Ψ defined in (5.12) *and $C = C(\bar{u}, \alpha, \beta, \nu, M, T) > 0$ independent of ε.*

Proof. Let $\mathcal{A} \subset \mathbb{R}^n$ denote a set such that $\bar{u}(x, t) \in \mathcal{A}$ for every (x, t). Then (H1), (H3-a), (H4) and (5.6) imply

$$\begin{aligned} d_1 &\le |u^\varepsilon - \bar{u}||G(u^\varepsilon) - G(\bar{u})| \le L_{\mathcal{A}}|u^\varepsilon - \bar{u}|^2 \\ d_2 &\le \alpha\, \|G(\bar{u})\|_{L^\infty} |u^\varepsilon - \bar{u}|^2 \\ \varepsilon d_3 &\le \varepsilon L_{\mathcal{A}}|u^\varepsilon - \bar{u}||(u^\varepsilon - \bar{u})_t| \le L_{\mathcal{A}}^2|u^\varepsilon - \bar{u}|^2 + \varepsilon^2|(u^\varepsilon - \bar{u})_t|^2 . \end{aligned} \tag{5.35}$$

Then, combining (5.12), (5.14) and (5.35), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \mathcal{G}(\bar u, u^\varepsilon)\,dx + \varepsilon \bar c \int_{\mathbb{R}} |u_x^\varepsilon - \bar u_x|^2 + |u_t^\varepsilon - \bar u_t|^2\,dx \le C\Big(\Psi(t) + \varepsilon^2\Big).$$

Integrating the above inequality, and using (5.13), we obtain

$$\Psi(t) \le C\Big(\Psi(0) + \int_0^t \Psi(s)\,ds + \varepsilon^2\Big)$$

and conclude (5.34) via the Gronwall lemma. □

6 Multidimensional case

In this section we state our results for multidimensional systems. The proofs of these theorems follow similar line of argument as the ones presented in the earlier parts of this article, and therefore are here omitted.

We define

$$\varphi(t) := \int_{\mathbb{R}^d} |u^\varepsilon|^2 + \varepsilon^2 |\mathrm{D}u^\varepsilon|^2 + \varepsilon^2 |u_t^\varepsilon|^2\,dx \tag{6.1}$$

$$\Psi(t) := \int_{\mathbb{R}} |\bar u - u^\varepsilon|^2 + \varepsilon^2 |\mathrm{D}\bar u - \mathrm{D}u^\varepsilon|^2 + \varepsilon^2 |(\bar u - u^\varepsilon)_t|^2\,dx \tag{6.2}$$

which are used in the next two subsections.

6.1 Systems with weakly dissipative source G, $d \ge 1$

The first result is the analog of Proposition 3.1.2 on stability.

Proposition 6.1.1. *Let $\{u^\varepsilon(x,t)\}$ be a family of smooth solutions to the system* (1.4) *on $\mathbb{R}^d \times [0,T]$. Suppose that $u \equiv u^\varepsilon$ decays fast at infinity and that:*

(a1) (H1) *holds true, and the positive definite, symmetric matrices A_j, $j = 1, \ldots, d$ are such that the subcharacteristic condition* (H2*) *is valid.*

(a2) The conditions (H3-a) *and* (H4) *for the source G hold true.*

Then for all $t \in [0,T]$

$$\varphi(t) + \varepsilon \int_{\mathbb{R}^d} R(u^\varepsilon(x,t))\,dx + \int_0^t \int_{\mathbb{R}^d} |\mathrm{D}\eta(u) G(u)|\,dxdt \le C\Big(\varphi(0) + \varepsilon \int_{\mathbb{R}^d} R(u^\varepsilon(x,0))\,dx\Big) \tag{6.3}$$

with $\varphi(t)$ defined in (6.1)*, $R(u)$ defined in* (H4) *and $C = C(A, \alpha, \beta) > 0$ independent of ε, T.*

The next theorem is the analog of compactness Theorem 4.1.3

Theorem 6.1.2. *Let $d \geq 1$. Let $\{u_\varepsilon\}$ be a family of smooth solutions of* (3.3) *on $\mathbb{R}^d \times [0,T]$ emanating from smooth initial data. The family $\{u_\varepsilon\}$ is assumed to decay fast at infinity. Let the hypotheses of Proposition 6.1.1 remain valid so that the stability estimate* (6.3) *holds true with $\eta - q$ entropy-entropy flux pair satisfying* (H1)*, and A_j, $j = 1, \dots, d$, symmetric, positive-definite matrices subject to* (H2*)*. Then for an entropy pair $\bar{\eta} - \bar{q}$ satisfying the growth conditions*

$$\|\bar{\eta}\|_{L^\infty}, \|\bar{q}\|_{L^\infty}, \|\mathrm{D}\bar{\eta}\|_{L^\infty}, \|\mathrm{D}^2\bar{\eta}\|_{L^\infty} \leq C$$

and

$$|\mathrm{D}\bar{\eta}(v)G(v)| \leq C\big(M - \mathrm{D}\eta(v)G(v)\big)\,, \quad \forall v \in \mathbb{R}^n$$

the family

$$\Big\{\partial_t \bar{\eta}(u^\varepsilon) + \sum_{j=1}^{d} \partial_{x_j} \bar{q}_j(u^\varepsilon)\Big\}_\varepsilon \text{ lies in a compact set of } H^{-1}_{loc}(\mathbb{R}^d \times [0,T]).$$

The next theorem is the analog of Theorem 5.3.1 on convergence.

Theorem 6.1.3. *Let $\bar{u}(x,t)$ be a smooth solution of the system* (3.1)*, defined on $\mathbb{R}^d \times [0,T]$. Let $\{u^\varepsilon\}$ be a family of smooth solutions of the relaxation system* (3.2) *on $\mathbb{R}^d \times [0,T]$. Suppose that both $\bar{u}$ and u^ε decay sufficiently fast at infinity and that:*

(a1) (H1) holds true and the positive definite, symmetric matrices A_j, $j = 1, \dots, d$ are such that the subcharacteristic condition (H2*) *is valid.*

(a2) For some $M > 0$

$$|D^2 F_j(u)| \leq M, \; j = 1, \dots, d, \quad |D^3\eta(u)| \leq M, \quad u \in \mathbb{R}^n.$$

(a3) The conditions (H3-a) *and* (H4) *for the source term G hold true.*

Then for all $t \in [0,T]$

$$\Psi(t) + \varepsilon \int_{\mathbb{R}^d} R(u^\varepsilon(x,t))\, dx \;\leq\; C\bigg(\Psi(0) + \varepsilon \int_{\mathbb{R}^d} R(u^\varepsilon(x,0))\, dx + \varepsilon\bigg)$$

with Ψ defined in (6.2)*, $R(u)$ defined in* (H4) *and $C = C(\bar{u}, \alpha, \beta, \nu, M, T) > 0$ independent of ε.*

If, in addition, $G \in C^2(\mathbb{R}^n)$ then for all $t \in [0,T]$

$$\Psi(t) + \varepsilon \int_{\mathbb{R}^d} \big[R^{rel}(\bar{u}, u^\varepsilon)\big](x,t)\, dx \;\leq\; C\bigg(\Psi(0) + \varepsilon \int_{\mathbb{R}^d} \big[R^{rel}(\bar{u}, u^\varepsilon)\big](x,0)\, dx + \varepsilon^2\bigg)$$

where

$$R^{rel}(\bar{u}, u^\varepsilon) := R(u^\varepsilon) - R(\bar{u}) - \mathrm{D}R(\bar{u})(u - \bar{u}) \geq 0\,.$$

6.2 Systems with general source G, $d \geq 1$

Proposition 6.2.1. *Let $\{u^\varepsilon(x,t)\}$ be a family of smooth solutions to* (1.4) *on* $\mathbb{R}^d \times [0,T]$*. Suppose that $u \equiv u^\varepsilon$ decays fast at infinity and that:*

(a1) (H1) holds true, and positive definite, symmetric matrices A_j, $j = 1, \dots, d$ are such that the subcharacteristic condition (H2*) *is valid.*

(a2) The condition (H3-b) *on the source G holds true.*

Then

$$\varphi(t) \leq C\,\varphi(0)\,, \quad t \in [0,T] \tag{6.4}$$

with φ defined in (6.1) *and $C = C(A, \alpha, \beta, T, L) > 0$ independent of ε and T.*

Theorem 6.2.2. *Let $d \geq 1$. Let $\{u_\varepsilon\}$ be a family of smooth solutions of* (3.3) *on* $\mathbb{R}^d \times [0,T]$ *emanating from smooth initial data. The family $\{u_\varepsilon\}$ is assumed to decay fast at infinity. Let the hypotheses of Proposition 6.2.1 remain valid so that the stability estimate* (6.4) *holds true with $\eta - q$ entropy-entropy flux pair satisfying* (H1)*, and A_j, $j = 1, \dots, d$, symmetric, positive-definite matrices subject to* (H2*)*. Then for an entropy pairs $\bar\eta - \bar q$ satisfying*

$$\|\bar\eta\|_{L^\infty}, \|\bar q\|_{L^\infty}, \|\mathrm{D}\bar\eta\|_{L^\infty}, \|\mathrm{D}^2\bar\eta\|_{L^\infty} \leq C$$

the family

$$\Big\{\partial_t \bar\eta(u^\varepsilon) + \sum_{j=1}^{d} \partial_{x_j} \bar q_j(u^\varepsilon)\Big\}_\varepsilon \text{ lies in a compact set of } H^{-1}_{loc}(\mathbb{R}^d \times [0,T]).$$

Theorem 6.2.3. *Let $\bar u(x,t)$ be a smooth solution of the system* (1.1)*, defined on* $\mathbb{R} \times [0,T]$*. Let $\{u^\varepsilon\}$ be a family of smooth solutions of the relaxation system* (1.2) *on* $\mathbb{R}^d \times [0,T]$*. Suppose that both $\bar u$ and u^ε decay sufficiently fast at infinity and that:*

(a1) (H1) holds true and the positive definite, symmetric matrix A, is such that the subcharacteristic condition (H2*) *is valid.*

(a2) *For some $M > 0$*

$$|\mathrm{D}^2 F_j(u)| \leq M,\ j = 1, \dots, d\,, \quad |\mathrm{D}^3 \eta(u)| \leq M, \quad u \in \mathbb{R}^n.$$

(a3) The condition (H3-b) for the source term G holds true.

Then

$$\Psi(t) \,\leq\, C\big(\Psi(0) + \varepsilon^2\big), \quad t \in [0,T]$$

with Ψ defined in (6.2) *and $C = C(\bar u, \alpha, \beta, \nu, M, T) > 0$ independent of ε.*

7 An alternative relaxation model

In this section we consider an alternative relaxation model for the system of hyperbolic balance laws (1.1) given by

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} \partial_{x_j} v_j = G(u) \\ \partial_t v_i + A_i \partial_{x_i} u = -\frac{1}{\varepsilon}\big(v_i - F_i(u)\big), \quad i = 1, \dots, d \end{cases} \tag{7.1}$$

with $u, v_i \in \mathbb{R}^n$ and A_i symmetric, positive definite matrix. Excluding v_i from the equation $(7.1)_1$ and assuming $G \in C^1(\mathbb{R}^n)$, we obtain

$$\partial_t u + \sum_{j=1}^{d} \partial_{x_j} F_j(u) = G(u) + \varepsilon\, \partial_t(G(u)) + \varepsilon\Big(\sum_{j=1}^{d} A_j u_{x_j x_j} - u_{tt}\Big) \tag{7.2}$$

that approximates the system of balance laws (1.1). We remark that the treatment of such relaxation systems presents several challenges. More specifically, the time derivative of the source term appear in the energy functional posing enormous difficulties in the analysis, an additional hypothesis is required for the establishment of stability (see (H7), Section 6), the issue of compactness is problematic. In the next two subsections we will study the stability and compactness properties of solutions to (7.2) in order to point out the advantages of the relaxation model (1.2). We restrict the analysis to weakly dissipative systems of hyperbolic balance laws equipped with strictly convex entropy, $d = 1$.

7.1 Weakly dissipative systems with strictly convex entropy, $d = 1$

The system (7.2) for $d = 1$ reads

$$\partial_t u + \partial_x F(u) = G(u) + \varepsilon\, \partial_t(G(u)) + \varepsilon(A u_{xx} - u_{tt}) \tag{7.3}$$

where A is symmetric, positive definite matrix. Following the arguments of Lemma 3.1.1 we obtain:

Lemma 7.1.1. *Suppose $u \equiv u^\varepsilon(x,t)$ is a smooth solution to the equation (7.3) on $\mathbb{R} \times [0,T]$, $\eta - q$ is the entropy-entropy flux pair of (3.1) and $\bar\alpha \in \mathbb{R}$ is fixed. Then, the following energy identity holds*

$$\begin{aligned} &\partial_t \Big[\eta(u + \varepsilon u_t) + \tfrac{1}{2}\varepsilon^2 \bar\alpha\, |u_t|^2 + \varepsilon^2 \bar\alpha\, u_x^\top A u_x \\ &\qquad + \varepsilon^2 u_t^\top \Big(\tfrac{1}{2}\bar\alpha \mathbf{I} - \int_0^1 \int_0^s \mathrm{D}^2 \eta(u + \varepsilon \tau u_t)\, d\tau ds \Big) u_t \Big] + \partial_x q(u) \\ &\quad + \varepsilon\bar\alpha \big| u_t + \mathrm{D}F(u) u_x \big|^2 + \varepsilon u_t^\top \Big(\bar\alpha I - \mathrm{D}^2\eta(u) \Big) u_t \\ &\quad + \varepsilon u_x^\top \Big(\mathrm{D}^2\eta(u) A - \bar\alpha \mathrm{D}F(u)^\top \mathrm{D}F(u) \Big) u_x \\ &= \partial_x \Big(\varepsilon \mathrm{D}\eta(u) A u_x + 2\varepsilon^2 \bar\alpha\, u_t^\top A u_x \Big) \\ &\quad + \mathrm{D}\eta(u) \Big(G(u) + \varepsilon \partial_t(G(u)) \Big) + 2\varepsilon\bar\alpha u_t^\top \Big(G(u) + \varepsilon \partial_t(G(u) \Big). \end{aligned} \tag{7.4}$$

In our further analysis we will employ the following elementary lemma.

Lemma 7.1.2 (Weak dissipation of a gradient). *Suppose $\eta(u)$ satisfies* (H1), *and $G(u) \in C^1$ satisfies* (H3-a), (H4). *Then*

$$-\mathrm{D}G(u) = \mathrm{D}^2 R(u) \geq 0 \quad \textit{for all} \quad u \in \mathbb{R}^n \,. \tag{7.5}$$

In addition, if $\tilde{\eta}(u) : \mathbb{R}^n \to \mathbb{R}$ satisfies $\mathrm{D}^2\tilde{\eta}(u) \geq 0$, then

$$-\mathrm{D}^2\tilde{\eta}(u)\mathrm{D}G(u) = \mathrm{D}^2\tilde{\eta}(u)\mathrm{D}^2 R(u) \geq 0 \quad \textit{for all} \quad u \in \mathbb{R}^n \,. \tag{7.6}$$

Proof. From (H3-a), (H4) it follows that

$$-z^\top \mathrm{D}^2\eta(u)\mathrm{D}G(u)\, z = z^\top \mathrm{D}^2\eta(u)\mathrm{D}^2 R(u)\, z \geq 0 \quad \text{for all} \quad u, z \in \mathbb{R}^n \,.$$

Fix $u \in \mathbb{R}^n$. Let (λ, v) be an eignepair of $\mathrm{D}^2 R(u)$. Then the above inequality implies

$$0 \leq v^\top \mathrm{D}^2\eta(u)\mathrm{D}^2 R(u)\, v = \lambda\big(v^\top \mathrm{D}^2\eta(u) v\big).$$

Recalling that $\mathrm{D}^2\eta(u) > 0$ and $v \neq 0$, we conclude that $\lambda \geq 0$. Hence all eigenvalues of $\mathrm{D}^2 R(u)$ are nonnegative. Since $\mathrm{D}^2 R(u)$ is a symmetric matrix this is equivalent to $\mathrm{D}^2 R(u) \geq 0$.

Next, suppose $\tilde{\eta}(u)$ is a strictly convex function. Then (7.6) follows from the fact that $\mathrm{D}^2\tilde{\eta}$, $\mathrm{D}^2 R$ are symmetric, nonnegative definite matrices. □

Remark 7.1.3. *Note that the requirement of uniform convexity for $\eta(u)$ in Lemma 7.1.2 can be replaced with strict convexity. Thus, when G is a gradient, weak dissipation of G with respect to some strictly convex entropy automatically yields weak dissipation with respect to any convex entropy, a property which is not in general satisfied.*

To establish the stability of solutions to (7.3) we will employ an additional hypothesis:

- Suppose there exists $S : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &\mathrm{D}\eta(u)\mathrm{D}G(u) = -\mathrm{D}S(u), \quad u \in \mathbb{R}^n \\ &S(u) \geq S(0) = 0\,. \end{aligned} \tag{H5}$$

Remark 7.1.4. *We note that the hypothesis* (H5) *is a severe assumption. It is motivated by the following observation. Suppose that the entropy $\eta(u) = \frac{1}{2}|u|^2$, and* (H3-a), (H4) *hold true. Then,*

$$\mathrm{D}\eta(u)\mathrm{D}G(u) = u^\top \mathrm{D}^2 R(u) = -\mathrm{D}S(u) \quad \textit{with} \quad S(u) = \mathrm{D}R(u)u - R(u)\,.$$

Moreover, by Lemma 7.1.2 it follows that $\mathrm{D}^2 R \geq 0$ and hence $S(u) \geq S(0) = 0$.

Proposition 7.1.5. *Let $\{u^\varepsilon(x,t)\}$ be a family of smooth solutions to the equation* (7.3) *defined on $\mathbb{R} \times [0,T]$. Suppose that $u \equiv u^\varepsilon$ decays fast at infinity and that:*

(a1) (H1) *holds true, and the positive definite, symmetric matrix A is such that the subcharacteristic condition* (H2) *is valid.*

(a2) The conditions (H3-a)*,* (H4)*, and* (H5) *for the source $G \in C^1(\mathbb{R}^n)$ hold true.*

Then for all $t \in [0, T]$

$$
\begin{aligned}
\varphi(t) + \varepsilon \int_{\mathbb{R}} \Big\{ S(u^\varepsilon(x,t)) + R(u^\varepsilon(x,t)) \Big\}\, dx + \int_0^t \int_{\mathbb{R}} |\mathrm{D}\eta(u) G(u)|\, dx dt \\
\le C\bigg(\varphi(0) + \varepsilon \int_{\mathbb{R}} \Big\{ S(u^\varepsilon(x,0)) + R(u^\varepsilon(x,0)) \Big\}\, dx \bigg)
\end{aligned}
\tag{7.7}
$$

with φ defined in (3.8) *and $C = C(A, \alpha, \beta) > 0$ independent of both ε and T.*

Proof. Using (H4), (H5) we rewrite the last four terms on the right-hand side of (7.4), with $\bar{\alpha} = \alpha$, as follows

$$
\begin{aligned}
&\Big(\mathrm{D}\eta(u) G(u) + \varepsilon \mathrm{D}\eta(u)\, \partial_t(G(u)) + 2\varepsilon u_t^\top G(u) + 2\varepsilon^2 \alpha u_t^\top \partial_t(G(u)) \Big) \\
&= \Big(\mathrm{D}\eta(u) G(u) - 2\varepsilon^2 \alpha\, u_t^\top \mathrm{D}^2 R(u)\, u_t \Big) - \varepsilon \partial_t \Big(S(u) + 2\alpha R(u) \Big) =: I_1 + I_2 .
\end{aligned}
\tag{7.8}
$$

Since $G \in C^1$, by (H1), (H3-a), (H4) and Lemma 7.1.2

$$
\alpha\, u_t^\top \mathrm{D}^2 R(u)\, u_t \ge 0 . \tag{7.9}
$$

Thus, by (H3-a) and (7.9) we conclude $I_1 \le 0$. Then, integrating the identity (7.4), with $\bar{\alpha} = \alpha$, and employing hypotheses (H1), (H2), (H3-a), (H4), (H5) along with (3.10), (3.12) and (7.8), we conclude (7.7). □

7.2 Compactness issues

Let $\{u_\varepsilon\}$ be a family of smooth solutions of (7.3) on $\mathbb{R} \times [0, T]$ emanating from smooth initial data. The family $\{u_\varepsilon\}$ is assumed to decay fast at infinity. Let the hypotheses of Lemma 7.1.5 remain valid so that the stability estimate (7.7) holds true with $\eta - q$ entropy-entropy flux pair satisfying (H1), and A a symmetric, positive-definite matrix subject to (H2). Let the entropy pairs $(\bar{\eta}, \bar{q})$ satisfy (4.1) and the growth condition holds (4.2). We now show that, in general, one may not expect compactness from the family $\{\partial_t \bar{\eta}(u^\varepsilon) + \partial_x \bar{q}(u^\varepsilon)\}$.

We follow the arguments of Theorem 4.1.2. Starting from (7.3) we obtain

$$
\begin{aligned}
\partial_t \bar{\eta}(u^\varepsilon) + \partial_x \bar{q}(u^\varepsilon) &= \varepsilon \partial_x \left(\mathrm{D}\bar{\eta}(u^\varepsilon) A u_x^\varepsilon \right) - \varepsilon \partial_t \left(\mathrm{D}\bar{\eta}(u^\varepsilon) u_t^\varepsilon \right) \\
&\quad - \varepsilon {u_x^\varepsilon}^\top \mathrm{D}^2 \bar{\eta}(u^\varepsilon) A u_x^\varepsilon + \varepsilon {u_t^\varepsilon}^\top \mathrm{D}^2 \bar{\eta}(u^\varepsilon) u_t^\varepsilon \\
&\quad + \mathrm{D}\bar{\eta}(u^\varepsilon) G(u^\varepsilon) + \varepsilon \mathrm{D}\bar{\eta}(u^\varepsilon) \mathrm{D}G(u^\varepsilon) u_t^\varepsilon \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 .
\end{aligned}
$$

As before from (4.1), (4.2) and (7.7), it follows that the terms I_1, I_2 lie in compact set of H^{-1}, and the terms I_3, I_4, I_5 are bounded in L^1. However, the term I_6 is, in general, neither in a bounded set of the space of measures nor in a compact set of H^{-1}: for the former one must have a control of $\varepsilon \mathrm{D}G$ in L^2, and for the latter a control of εG in L^2, which follows from the relation

$$
I_6 = \partial_t \big(\mathrm{D}\bar{\eta}(u^\varepsilon) \varepsilon G(u^\varepsilon) \big) - \big(\mathrm{D}^2 \bar{\eta}(u^\varepsilon) u_t^\varepsilon \big)^\top \varepsilon G(u^\varepsilon) .
$$

The stability estimate (7.7), however, does not provide such bounds. Thus, Murat's lemma is not applicable and the issue of compactness appears problematic.

8 Applications

8.1 Elasticity system

Consider the relaxation of the (isothermal/isentropic) elasticity system with a source term:

$$\begin{bmatrix} u \\ v \end{bmatrix}_t - \begin{bmatrix} v \\ \sigma(u) \end{bmatrix}_x = G(u,v) = \begin{bmatrix} 0 \\ g(u,v) \end{bmatrix}. \tag{8.1}$$

In the context of gas flow, u is specific volume ($u = 1/\rho$), thus constrained by $u > 0$. In the context of the thermoelastic bar, u is the *strain*, likewise constrained by $u > 0$. Finally, in the context of shearing motion, u is shearing, which may take both positive and negative values. In the gas case, it is traditional to use the pressure $p = -\sigma$, instead of σ.

In the present context, the stress $\sigma(u)$ is assumed to satisfy

$$\sigma(0) = 0 \quad \text{and} \quad 0 < \gamma < \sigma'(u) < \Gamma \quad \text{for all} \quad u \in \mathbb{R}^n. \tag{8.2}$$

We assume that $g(u,v)$, with $g(0,0) = 0$, satisfies one of the following:

(i) *Either* g is independent of u, that is $g(u,v) = g(v)$, and satisfies

$$\big(g(v) - g(\bar v)\big)\big(v - \bar v\big) \le 0, \quad \forall v, \bar v \in \mathbb{R} \tag{8.3}$$

which corresponds to a *frictional damping.*

(ii) *or* for every compact set $\mathcal{A} \subset \mathbb{R}^2$ there exists $L_{\mathcal{A}} > 0$ such that

$$|g(u,v) - g(\bar u, \bar v)| \le L_{\mathcal{A}}\big(|u - \bar u| + |v - \bar v|\big) \tag{8.4}$$

for all $(u,v) \in \mathbb{R}^2$, $(\bar u, \bar v) \in \mathcal{A}$.

The system (8.1) is equipped with the entropy - entropy flux pair $\bar\eta, \bar q$ given by

$$\eta(u,v) = \tfrac{1}{2}v^2 + \Sigma(u), \quad q(u,v) = -\sigma(u)v \quad \text{with} \quad \Sigma(u) := \int_0^u \sigma(\tau)\, d\tau\,. \tag{8.5}$$

For the system (8.1) the second order relaxation system (3.3) reads

$$\begin{bmatrix} u \\ v \end{bmatrix}_t - \begin{bmatrix} v \\ \sigma(u) \end{bmatrix}_x = \begin{bmatrix} 0 \\ g(u,v) \end{bmatrix} + \varepsilon\left(A \begin{bmatrix} u \\ v \end{bmatrix}_{xx} - \begin{bmatrix} u \\ v \end{bmatrix}_{tt} \right). \tag{8.6}$$

Observe that by (8.2) and (8.5) hypotheses (H1)-(H2) are satisfied with

$$\alpha = \max(2\Gamma, 1), \quad \beta = \min(\gamma, 1), \quad A = 2\alpha \mathbf{I}\,. \tag{8.7}$$

Suppose that (8.3) holds true. Then

$$\big(\mathrm{D}\eta(u,v) - \eta(\bar u, \bar v)\big)\big(G(u,v) - G(\bar u, \bar v)\big) = (v - \bar v)(g(v) - g(\bar v)) \le 0$$

and hence G satisfies (H3-a). Furthermore, setting

$$R(u,v) := -\int_0^v g(\theta)\,d\theta \quad \text{we get} \quad G(u,v) = -\mathrm{D}R(u,v)\,, \quad R(0,0)=0$$

which gives (H4). Thus, one may employ Proposition 3.1.2 to obtain the stability estimate (3.9) and Theorem 5.3.1 to obtain the error estimate (8.6) that holds before the formation of shocks.

Similarly, suppose (8.4) holds true. Then clearly G satisfies (H3-b) and one may use Proposition 3.1.3 and Theorem 5.4.1 to obtain the stability estimate (3.13) and the error estimate (8.6), respectively.

The stability estimates (3.9) and (3.13) suffice to apply the L^p theory of compensated compactness. In the spirit of [21, Theorem 1] we prove the following convergence theorem.

Theorem 8.1.1. *Let $\sigma(u)$ satisfy (8.2) and suppose also*

$$(u-u_0)g''(u) \neq 0 \ \text{ for } \ u \neq u_0 \ \text{ and } \ g'',\, g''' \in L^2 \bigcap L^\infty. \tag{8.8}$$

Let $(u^\varepsilon, v^\varepsilon)$ be a family of smooth solutions of (8.1) defined on $\mathbb{R}\times[0,T)$ emanating from smooth initial data subject to ε-independent bounds

$$\varphi(0) = \int_{\mathbb{R}} \Big(u_0^{\varepsilon\,2} + v_0^{\varepsilon\,2} \Big) dx + \varepsilon^2 \int_{\mathbb{R}} \Big(|u_{0x}^\varepsilon|^2 + |u_{0t}^\varepsilon|^2 \Big) dx \;\leq\; C_0.$$

Let A be a symmetric, positive definite matrix satisfying (8.7) and let $g(u,v)$ satisfy either (8.3) or (8.4). Then, along a subsequence if necessary,

$$u^\varepsilon \to u\,,\; v^\varepsilon \to v\,, \ \text{ a.e. } (x,t) \ \text{ and in } \ L^p_{loc}(\mathbb{R}\times(0,T)), \quad \text{for } \ p<2.$$

Proof. Let $(u^\varepsilon, v^\varepsilon)$ be a family of solutions to (8.6). The proof uses the theory of compensated compactness [27]. Typically, in such proofs, the goal is to control the dissipation measure and to show that

$$\Big\{ \partial_t \bar\eta(u^\varepsilon, v^\varepsilon) + \partial_x \bar q(u^\varepsilon, v^\varepsilon) \Big\}_\varepsilon \ \text{lies in a compact set of} \ \ H^{-1}_{loc}(\mathbb{R}\times(0,T)) \tag{8.9}$$

for a class of entropy-entropy flux pairs $\bar\eta - \bar q$ for the equations of elasticity. In the presence of uniform L^∞-bounds, the theorem of DiPerna [13] guarantees compactness of approximate solutions and implies that, along a subsequence, $u^\varepsilon \to u$ and $v^\varepsilon \to v$ a.e. (x,t).

In the present case L^1-estimates are only available in the special case that A is a multiple of the identity matrix (see [23]) and, in view of (3.9) and (3.13), the natural stability framework is in the energy norm. Convergence of viscosity approximations to the equations of elastodynamics in the energy framework is carried out in Shearer [25] (for the genuine-nonlinear case) and Serre-Shearer [26] (for loss of genuine nonlinearity at one point). In [25] two classes of entropies, with growth controlled by the wave-speeds at infinity, are constructed ([25, Lemma 2]) for which Tartar's commutation relation is justified (see [25, Lemma 2]) and are used to show that the support of the (generalized) Young measure is a point mass ([25, Lemma 7, Theorem 1-(iii)]). When $\sigma(u)$ has one inflection point, the reduction of the Young measure is performed in [26, Lemma 3] and [26, Section 5].

To ensure the dissipation estimate, we employ the growth assumption (8.2), the subcharacteristic condition $(8.7)_3$ and the assumption on the source (8.3), (8.4). Then it suffices to establish (8.9) for entropy pairs $\bar{\eta} - \bar{q}$ satisfying

$$\|\bar{\eta}\|_{L^\infty}\,, \|\bar{q}\|_{L^\infty}\,, \|\mathrm{D}\bar{\eta}\|_{L^\infty}\,, \|\mathrm{D}^2\bar{\eta}\|_{L^\infty} \le C\,. \tag{8.10}$$

This class of entropy pairs contains (under the assumption (8.2)) the test-pairs that are used in [25, 26] in order to prove the reduction of the generalized Young measure to a point mass and to show strong convergence in L^p_{loc} for $p < 2$. Hypothesis (8.8) reflects the assumptions needed in those works. To complete the proof, we show that (8.9) holds for entropy-entropy flux pair $\bar{\eta} - \bar{q}$ satisfying (8.10).

First suppose that $g(u, v)$ satisfies (8.3). Then by (8.5)

$$|\mathrm{D}\bar{\eta}(u, v)G(u, v)| = |\bar{\eta}_v(u, v)g(v)| \le C\big(M + \mathrm{D}\eta G(u, v)\big)\,, \quad M = \max_{|v|\le 1} |g(v)|\,. \tag{8.11}$$

The inequality (8.11) is the analog of the condition (4.2) used in Theorem 4.1.2 (in which case, the condition (4.2) is *automatically* satisfied for the elasticity system (8.1)). Then by (8.10), (8.11), and Theorem 4.1.2 we conclude (8.9).

Suppose now that $g(u, v)$ satisfies (8.4). Then G satisfies (H3-b) and hence from (8.10), (8.11), and Theorem 4.1.3 we obtain (8.9).

□

We remark that the alternative relaxation system (7.3) reads

$$\begin{bmatrix} u \\ v \end{bmatrix}_t - \begin{bmatrix} v \\ \sigma(u) \end{bmatrix}_x = \begin{bmatrix} 0 \\ g(u, v) \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ g(u, v) \end{bmatrix}_t + \varepsilon\Big(A \begin{bmatrix} u \\ v \end{bmatrix}_{xx} - \begin{bmatrix} u \\ v \end{bmatrix}_{tt} \Big)\,. \tag{8.12}$$

Then setting

$$S(u, v) := -g(v)v - R(u, v) \quad \text{we get} \quad \mathrm{D}\eta(u, v)\mathrm{D}G(u, v) = vg'(v) = -\mathrm{D}S(u, v),\ S(0, 0) = 0$$

which gives (H5). Thus, the stability of solutions to (8.12) follows from Proposition 7.1.5.

8.2 Isentropic combustion model

The governing equations for chemical reaction from *unburnt* gases to *burnt* gases in certain physical regimes (in Lagrangian coordinates) read [7]:

$$\begin{aligned} &\partial_t v - \partial_x u = 0 \\ &\partial_t u + \partial_x (P(v, s, Z)) = 0 \\ &\partial_t \big(E(v, s, Z) + \tfrac{1}{2}u^2 + qZ\big)_t + \partial_x (uP(v, s, Z)) = r \\ &\partial_t Z + K\varphi(\Theta(v, s, Z))Z = 0 \end{aligned} \tag{8.13}$$

The state of the gas is characterized by the macroscopic variables: the specific volume $v(x, t)$, the velocity field $u(x, t)$, the entropy $s(x, t)$ and the mass fraction of the reactant $Z(x, t)$, whereas the physical properties of the material are reflected through appropriate constitutive relations which relate the pressure $P(v, s, Z)$, internal energy $E(v, s, Z)$ with the macroscopic variables. Here, and in what follows, q represents the difference in the heats between the reactant and the product, K denotes the rate of the reactant, whereas $\varphi(\theta) \geqslant 0$ is the reaction function. The function $r(x,t)$ represents a source term (additional radiating heat density).

In this section we address the problem of relaxation to the *isentropic combustion* model

$$\begin{bmatrix} v \\ u \\ Z \end{bmatrix}_t + \begin{bmatrix} -u \\ P(v,Z) \\ 0 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \\ -K\varphi(\Theta(v,Z)) \end{bmatrix} \tag{8.14}$$

that arises naturally from the system (8.13) by externally regulating r to ensure $s = s_0$ [10]; in the sequel we suppress the variable s and use the notation

$$P(v,Z) := P(v,s_0,Z), \quad \Theta(v,Z) := \Theta(v,s_0,Z).$$

We impose the following requirements on P, Θ (see [16] for the motivation):

$(a1)$ Motivated by the physical property $\partial_v P < 0$ we assume that

$$0 < \gamma < -\partial_v P(v,Z) < \Gamma, \quad v \in \mathbb{R}, Z \in [0,1].$$

$(a2)$ There exists $\bar{C} > 0$ such that

$$\Big|\int_0^v P_{ZZ}(\tau,Z)\, d\tau\Big| < \bar{C}, \qquad |\partial_Z P(v,Z)| < \bar{C}, \quad v \in \mathbb{R}, Z \in [0,1].$$

$(a3)$ The composition $\varphi \circ \Theta$ of the rate and constitutive temperature functions satisfies for some $L > 0$

$$\big|\varphi(\Theta(v,Z)) - \varphi(\Theta(\bar{v},\bar{Z}))\big| \leq L|(v,Z) - (\bar{v},\bar{Z})| \tag{8.15}$$

for all $(v,Z), (\bar{v},\bar{Z}) \in \mathbb{R} \times [0,1]$.

Under $(a1)$-$(a3)$ the system (8.14) admits an entropy-entropy flux pair $\bar{\eta}, \bar{q}$ of the form:

$$\eta(v,u,Z) = \frac{1}{2}u^2 - \left(\int_0^v P(\tau,Z)\, d\tau\right) + B(Z)\,, \quad q(v,u,Z) = P(v,Z)u$$

with $B(Z)$ an *arbitrary function*. To ensure the convexity of η we assume, in addition to (a1)-(a3), that

$$B''(Z) > \Big(1 + \frac{2}{\Gamma}\bar{C}^2 + \bar{C}\Big), \quad Z \in [0,1]\,,$$

in which case

$$0 < \min\big(\frac{\gamma}{2}, 1\big) \leq \mathrm{D}^2\eta(v,u,Z) \leq \max\Big(1, \Gamma + \bar{C}, \frac{2}{\Gamma}\bar{C}^2 + 2\bar{C}\Big). \tag{8.16}$$

For the system (8.14) the second order relaxation system (3.3) reads

$$\begin{bmatrix} v \\ u \\ Z \end{bmatrix}_t + \begin{bmatrix} -u \\ P(v,Z) \\ 0 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \\ -K\varphi(\Theta(v,Z)) \end{bmatrix} + \varepsilon\left(A \begin{bmatrix} u \\ v \\ Z \end{bmatrix}_{xx} - \begin{bmatrix} u \\ v \\ Z \end{bmatrix}_{tt} \right)$$

Clearly by (8.16) hypotheses (H1) and (H2) are satisfied with

$$\alpha = \max\left(1, \Gamma + \bar{C}, \frac{2}{\Gamma}\bar{C}^2 + 2\bar{C}\right), \quad \beta = \min\left(\frac{\gamma}{2}, 1\right), \quad A = 2\alpha\mathbf{I}$$

while, in view of (8.15), the source G satisfies (H3-b). Thus, one may use Proposition 3.1.3, Theorem 4.1.3 and Theorem 5.4.1 to conclude about the stability, compactness and the error estimates before the formation of shocks, respectively.

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References

[1] Ch. Arvanitis, Ch. Makridakis, A.E. Tzavaras, Stability and Convergence of a Class of Finite Element Schemes for Hyperbolic Conservation Laws, *SIAM J. on Numer. Analysis* (2005), 42, 4, 1357-1393.

[2] D. Aregba-Driollet, R. Natalini, Convergence of relaxation schemes for conservation laws, *Appl. Anal.* (1996), 61, 163-193.

[3] D. Aregba-Driollet, R. Natalini, Discrete kinetic schemes for multi-dimensional conservation laws, *SIAM J. Numer. Anal.* (2000), 37, 6, 1973-2004.

[4] Y. Brenier, *Convergence of the Vlasov-Poisson system to the incompressible Euler equations*, Comm. Partial Diff. Equations, 25, 737-754, 2000.

[5] Y. Brenier, R. Natalini and M. Puel, *On the relaxation approximation of the incompressible Navier-Stokes equations* , Proc. Amer. Math. Soc., 132, 1021-1028, 2004.

[6] F. Berthelin and A. Vasseur, *From Kinetic Equations to Multidimensional Isentropic Gas Dynamics Before Shocks*, SIAM J. Math. Anal. 36, 1807-1835, 2005.

[7] G.-Q. Chen, D. Hoff and K. Trivisa, Global Solutions to a Model for Exothermically Reacting, Compressible Flows with Large Discontinuous Initial Data. *Arch. Ration. Mech. Anal.* (2003), 166, 321-358.

[8] G.-Q. Chen, C.D. Levermore and T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.* (1994), 47, 789-830.

[9] F. Coquel, B. Perthame, Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, *SIAM J. Numer. Anal.* (1998), 35 , 2223-2249.

[10] C.M. Dafermos, *The second law of thermodynamics and stability.* Arch. Rational Mech. Anal. (1979), 70, 167-179.

[11] C. M. Dafermos, Hyperbolic systems of balance laws with weak dissipation, *J. Hyp. Diff. Equations* (2006), 3, 505–527.

[12] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics* (Third edition. Grundlehren der Mathematischen Wissenschaften, 325. Springer-Verlag, Berlin, 2010).

[13] R. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Analysis* (1983), 82, 75-100.

[14] S. Jin and X. Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.* (1995), 48, 235-277.

[15] B. Hanouzet, R. Natalini, Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Arch. Rational Mech. Anal.,* (2003), 169, 89-117.

[16] A. Miroshnikov, K. Trivisa, Relative entropy in hyperbolic relaxation for balance laws, in *Commun. Math. Sci.* (to appear).

[17] R. Natalini, Convergence to equilibrium for the relaxation approximations of conservation laws, *Comm. Pure Appl. Math.* 49 (1996), 795-823.

[18] B. Perthame, An introduction to kinetic schemes for gas dynamics. *An introduction to Recent Developments in Theory and Numerics for Conservation Laws* (1998), D. Kroener, M. Ohlberger and C. Rhode, eds, Lecture Notes in Comp. Science and Engin., 5 Springer, Berlin, 1-27.

[19] T. Katsaounis, C. Makridakis, Relaxation models and finite element schemes for the shallow water equations. *Hyperbolic problems: theory, numerics, applications* (2003), 621-631, Springer, Berlin.

[20] P.G. LeFloch *Hyperbolic Systems of Conservation Laws: The Theory of Classical and Nonclassical Shock Waves*, Birkhauser, 2002 edition.

[21] L. Gosse, A. Tzavaras, Convergence of relaxation schemes to the equations of elastodynamics. *Math. Comp.* (2001), 70, 234, 555-577.

[22] F. Murat, L'injection du cone positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* (1981), 60 , 309-322.

[23] D. Serre, Relaxation semi-linéaire et cinétique des systémes de lois de conservation, *Ann. Inst. H. Poincaré, Anal. Non Linéaire.* (2000), 17, 2, 169-192

[24] S. Serre, *Systems of Conservation Laws 2: Geometric Structures, Oscillations, and Initial-Boundary Value.* Cambridge University Press 2000.

[25] J. Shearer, Global existence and compactness in L^p for the quasilinear wave equation, *Comm. Partial Diff. Eqs.* (1994), 19, 1829-1877.

[26] D. Serre, J. Shearer, Convergence with physical viscosity for nonlinear elasticity, (1993). (preprint).

[27] L. Tartar, Compensated Compactness and applications to partial differential equations. *Nonlinear Analysis and Mechanics - Heriot-Watt symposium - vol IV,* R.J. Knops, ed., Pitman Research notes in Mathematics, New York (1979), pp. 136-192.

[28] A. Tzavaras, Viscosity and relaxation approximations for hyperbolic systems of conservation laws. *An introduction to Recent Developments in Theory and Numerics for Conservation Laws*, D. Kroener, M. Ohlberger and C. Rhode, eds, Lecture Notes in Comp. Science and Engin., 5 Springer, Berlin, 1998, 73-122.

[29] A. Tzavaras, Materials with internal variables and relaxation to conservation laws. *Arch. Ration. Mech. Anal.* 146, 2, (1999), 129155.