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Convergence of Variational Approximation Schemes for Elastodynamics with Polyconvex Energy

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Abstract. We consider a variational scheme developed by S. Demoulini, D. M. A. Stuart and A. E. Tzavaras [Arch. Ration. Mech. Anal. 157 (2001), 325–344] that approximates the equations of three dimensional elastodynamics with polyconvex stored energy. We establish the convergence of the time-continuous interpolates constructed in the scheme to a solution of polyconvex elastodynamics before shock formation. The proof is based on a relative entropy estimation for the time-discrete approximants in an environment of L^p-theory bounds, and provides an error estimate for the approximation before the formation of shocks.

Keywords. Nonlinear elasticity, polyconvexity, variational approximation scheme

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1. Introduction

The equations of nonlinear elasticity are the system

$$y_{tt} = \operatorname{div} \frac{\partial W}{\partial F}(\nabla y)$$

where $y : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ stands for the motion, and we have employed the constitutive theory of hyperelasticity, i.e. the Piola-Kirchhoff stress tensor S is expressed as the gradient, $S(F) = \frac{\partial W}{\partial F}(F)$, of a stored energy function $W(F)$. The equations (1) are often recast as a system of conservation laws,

$$\begin{aligned} \partial_t v_i &= \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}}(F) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i, \end{aligned} \tag{1}$$

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for the velocity $v = \partial_t y$ and the deformation gradient $F = \nabla y$. The differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0$$

are propagated from the kinematic equation $(1)_2$ and are an involution, [7].

The requirement of frame indifference imposes that $W(F) : M_+^{3\times 3} \to [0,\infty)$ be invariant under rotations. This renders the assumption of convexity of W too restrictive [15], and convexity has been replaced by various weaker conditions familiar from the theory of elastostatics, see [1–3] for a recent survey. A commonly employed assumption is that of polyconvexity, postulating that

$$W(F) = G \circ \Phi(F)$$

where $\Phi(F) := (F, \operatorname{cof} F, \det F)$ is the vector of null-Lagrangians and $G = G(F, Z, w) = G(\Xi)$ is a convex function of $\Xi \in \mathbb{R}^{19}$; this encompasses certain physically realistic models [4, Section 4.9, 4.10]. Starting with the work of Ball [1], substantial progress has been achieved for handling the lack of convexity of W within the existence theory of elastostatics.

For the elastodynamics system local existence of classical solutions has been established in [6], [8, Theorem 5.4.4] for rank-1 convex stored energies, and in [8, Theorem 5.5.3] for polyconvex stored entropies. The existence of global weak solutions is an open problem, except in one-space dimension, see [12]. Construction of entropic measure valued solutions has been achieved in [9] using a variational approximation method associated to a time-discretized scheme. Various uniqueness results of smooth solutions in the class of entropy weak and even dissipative measure valued solutions are available for the elasticity system [7, 8, 10, 13].

The objective of the present work is to show that the approximation scheme of [9] converges to the classical solution of the elastodynamics system before the formation of shocks. To formulate the problem we outline the scheme in [9] and refer to Section 2 for a detailed presentation. The null-Lagrangians $\Phi^A(F)$, $A = 1, \dots, 19$ satisfy [14] the nonlinear transport identities

$$\partial_t \Phi^A(F) = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right) .$$

This allows to view the system (1) as constrained evolution of the extended system

$$\begin{aligned} \partial_t v_i &= \partial_\alpha \Big(\frac{\partial G}{\partial \Xi_A}(\Xi) \, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \Big) \\ \partial_t \Xi_A &= \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \, v_i \Big). \end{aligned} \tag{2}$$

The extension (2) has the properties: if $F(\cdot, 0)$ is a gradient and $\Xi(\cdot, 0) = \Phi(F(\cdot, 0))$, then $F(\cdot, t)$ remains a gradient and $\Xi(\cdot, t) = \Phi(F(\cdot, t))$ for all t. The extended system is endowed with the entropy identity

$$\partial_t\Big(\frac{|v|^2}{2}+G(\Xi)\Big)-\partial_\alpha\Big(v_i\,\frac{\partial G}{\partial \Xi_A}(\Xi)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\Big)=0$$

the entropy is convex and the system (2) is thus symmetrizable.

For periodic solutions v, Ξ (on the torus $\mathbb{T}^3$) a variational approximation method based on the time-discretization of (2) is proposed in [9]: Given a time-step $h > 0$ and initial data (v^0, Ξ^0) the scheme provides the sequence of iterates (v^j, Ξ^j), $j \geqslant 1$, by solving

$$\left.\begin{aligned}\frac{v_i^j - v_i^{j-1}}{h} &= \partial_\alpha\Big(\frac{\partial G}{\partial \Xi_A}(\Xi^j)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}\,\big(F^{j-1}\big)\Big)\\ \frac{(\Xi^j - \Xi^{j-1})_A}{h} &= \partial_\alpha\Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}}\,\big(F^{j-1}\big)\,v_i^j\Big).\end{aligned}\right. \quad \text{in } \mathcal{D}'(\mathbb{T}^3) \tag{3}$$

This problem is solvable using variational methods and the iterates (v^j, Ξ^j) give rise to a time-continuous approximate solution $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$. It is proved in [9] that the approximate solution generates a measure-valued solution of the equations of polyconvex elastodynamics.

In this work we consider a smooth solution of the elasticity system $\bar\Theta = (\bar V, \bar\Xi)$ defined on $[0,T]\times\mathbb{T}^3$ and show that the approximate solution $\Theta^{(h)}$ constructed via the iterates (v^j, Ξ^j) of (3) converges to $\bar\Theta = (\bar V, \bar\Xi)$ at a convergence rate $O(h)$. The method of proof is based on the relative entropy method developed for convex entropies in [5, 11] and adapted for the system of polyconvex elasticity in [13] using the embedding to the system (2). The difference between $\Theta^{(h)}$ and $\bar\Theta$ is controlled by monitoring the evolution of the relative entropy

$$\eta^r = \frac{1}{2}|V^{(h)} - \bar V|^2 + G(\Xi^{(h)}) - G(\bar\Xi) - \nabla G(\bar\Xi)(\Xi^{(h)} - \bar\Xi)\,.$$

We establish control of the function

$$\mathcal{E}(t) := \int_{\mathbb{T}^3}\Big((1+|F^{(h)}|^{p-2}+|\bar F|^{p-2})|F^{(h)}-\bar F|^2 + |\Theta^{(h)}-\bar\Theta|^2\Big)\,dx$$

and prove the estimation

$$\mathcal{E}(t) \leqslant C\Big(\mathcal{E}(0)+h\Big), \quad t\in[0,T]$$

which provides the result. There are two novelties in the present work: (a) In adapting the relative entropy method to the subject of time-discretized approximations. (b) In employing the method in an environment where L^p-theory needs to be used for estimating the relative entropy.

This work is a first step towards implementing a finite element method based on the variational approximation. To do that, one has to devise appropriate finite element spaces that preserve the involution structure. This is the subject of a future work.

The paper is organized as follows. In Section 2 we present the variational approximation scheme and state the Main Theorem. In Section 3 we derive the relative entropy identity (19) and, finally, in Section 4 we carry out the cumbersome estimations for the terms in the relative entropy identity and conclude the proof of Main Theorem via Gronwall's inequality.

2. The variational approximation scheme and statement of the Main Theorem

We assume that the stored energy $W : M^{3\times 3}_{+} \to \mathbb{R}$ is *polyconvex*:

$$W(F) = G \circ \Phi(F) \tag{4}$$

with

$$G = G(\Xi) = G(F, Z, w) : M^{3\times 3} \times M^{3\times 3} \times \mathbb{R} \cong \mathbb{R}^{19} \to \mathbb{R}$$

uniformly convex and

$$\Phi(F) = (F, \operatorname{cof} F, \det F). \tag{5}$$

Assumptions. We work with periodic boundary conditions, i.e. the spatial domain Ω is taken to be the three dimensional torus $\mathbb{T}^3$. The indices $i, \alpha, \dots$ generally run over $1, \dots, 3$ while $A, B, \dots$ run over $1, \dots, 19$. We use the notation $L^p = L^p(\mathbb{T}^3)$ and $W^{1,p} = W^{1,p}(\mathbb{T}^3)$. Finally, we impose the following convexity and growth assumptions on G:

(H1) $G \in C^3(M^{3\times 3} \times M^{3\times 3} \times \mathbb{R}; [0, \infty))$ is of the form

$$G(\Xi) = H(F) + R(\Xi) \tag{6}$$

with $H \in C^3(M^{3\times 3}; [0, \infty))$ and $R \in C^3(M^{3\times 3} \times M^{3\times 3} \times \mathbb{R}; [0, \infty))$ strictly convex satisfying

$$\kappa |F|^{p-2} |z|^2 \leqslant z^T \nabla^2 H(F) z \leqslant \kappa' |F|^{p-2} |z|^2, \quad \forall z \in \mathbb{R}^9$$

and $\gamma I \leqslant \nabla^2 R \leqslant \gamma' I$ for some fixed $\gamma, \gamma', \kappa, \kappa' > 0$ and $p \in [6, \infty)$.

(H2) $G(\Xi) \geqslant c_1 |F|^p + c_2 |Z|^2 + c_3 |w|^2 - c_4$.

(H3) $G(\Xi) \leqslant c_5 (|F|^p + |Z|^2 + |w|^2 + 1)$.

(H4) $|G_F|^{\frac{p}{p-1}} + |G_Z|^{\frac{p}{p-2}} + |G_w|^{\frac{p}{p-3}} \leqslant c_6 \left(|F|^p + |Z|^2 + |w|^2 + 1\right).$

(H5) $\left| \frac{\partial^3 H}{\partial F_{i\alpha} \partial F_{ml} \partial F_{rs}} \right| \leqslant c_7 |F|^{p-3}$ and $\left| \frac{\partial^3 R}{\partial \Xi_A \partial \Xi_B \partial \Xi_D} \right| \leqslant c_8.$

Notations. To simplify notation we write

$$G_{,A}(\Xi) = \frac{\partial G}{\partial \Xi_A}(\Xi), \qquad R_{,A}(\Xi) = \frac{\partial R}{\partial \Xi_A}(\Xi),$$
$$H_{,i\alpha}(F) = \frac{\partial H}{\partial F_{i\alpha}}(F), \qquad \Phi^A_{,i\alpha}(F) = \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F).$$

In addition, for each $i, \alpha = 1, 2, 3$ we set

$$g_{i\alpha}(\Xi, F^*) = \frac{\partial G}{\partial \Xi_A}(\Xi)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^*), \quad F^* \in \mathbb{R}^9,\ \Xi \in \mathbb{R}^{19} \tag{7}$$

(where we use the summation convention over repeated indices) and denote the corresponding fields $g_i : \mathbb{R}^{19} \times \mathbb{R}^9 \to \mathbb{R}^3$ by

$$g_i(\Xi, F^*) := (g_{i1}, g_{i2}, g_{i3})(\Xi, F^*).$$

2.1. Time-discrete variational scheme. The equations of elastodynamics (1) for polyconvex stored-energy (4) can be expressed as a system of conservation laws,

$$\begin{aligned} \partial_t v_i &= \partial_\alpha\left(\frac{\partial G}{\partial \Xi_A}(\Phi(F))\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \\ \partial_t F_{i\alpha} &= \partial_\alpha v_i \end{aligned} \tag{8}$$

which is equivalent to (1) subject to differential constrains

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \tag{9}$$

that are an involution [7]: if they are satisfied for $t = 0$ then (8) propagates (9) to satisfy for all times. Thus the system (8) is equivalent to systems (1) whenever $F(\cdot, 0)$ is a gradient.

The components of $\Phi(F)$ defined by (5) are null-Lagrangians and satisfy

$$\partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\right) = 0, \quad A = 1, \dots, 19$$

for any smooth $u(x) : \mathbb{R}^3 \to \mathbb{R}^3$. Therefore, if (v, F) are smooth solutions of (8), the null-Lagrangians $\Phi^A(F)$ satisfy the transport identities [9]

$$\partial_t \Phi^A(F) = \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i\right), \quad \forall F \text{ with } \partial_\beta F_{i\alpha} = \partial_\alpha F_{i\beta}. \tag{10}$$

Due to the identities (10) the system of polyconvex elastodynamics (8) can be embedded into the enlarged system [9]

$$\begin{aligned} \partial_t v_i &= \partial_\alpha\left(\frac{\partial G}{\partial \Xi_A}(\Xi)\,\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \\ \partial_t \Xi_A &= \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i\right). \end{aligned} \tag{11}$$

The extension has the following properties:

(E 1) If $F(\cdot,0)$ is a gradient then $F(\cdot,t)$ remains a gradient for all t.

(E 2) If $F(\cdot,0)$ is a gradient and $\Xi(\cdot,0) = \Phi(F(\cdot,0))$, then $F(\cdot,t)$ remains a gradient and $\Xi(\cdot,t) = \Phi(F(\cdot,t))$ for all t. In other words, the system of polyconvex elastodynamics can be viewed as a constrained evolution of (11).

(E 3) The enlarged system admits a convex entropy

$$\eta(v,\Xi) = \tfrac{1}{2}|v|^2 + G(\Xi), \quad (v,\Xi) \in \mathbb{R}^{22} \tag{12}$$

and thus is symmetrizable (along the solutions that are gradients).

Based on the time-discretization of the enlarged system (11) S. Demoulini, D. M. A. Stuart and A. E. Tzavaras [9] developed a variational approximation scheme which, for the given initial data

$$\Theta^0 := (v^0, \Xi^0) = (v^0, F^0, Z^0, w^0) \in L^2 \times L^p \times L^2 \times L^2$$

and fixed $h > 0$, constructs the sequence of successive iterates

$$\Theta^j := (v^j, \Xi^j) = (v^j, F^j, Z^j, w^j) \in L^2 \times L^p \times L^2 \times L^2, \quad j \geqslant 1$$

with the following properties (see [9, Lemma 1, Corollary 2]):

(P 1) The iterate (v^j, Ξ^j) is the unique minimizer of the functional

$$\mathcal{J}(v,\Xi) = \int_{\mathbb{T}^3} \Big(\tfrac{1}{2}|v - v^{j-1}|^2 + G(\Xi) \Big)\, dx$$

over the weakly closed affine subspace

$$\mathcal{C} = \Bigg\{ (v,\Xi) \in L^2 \times L^p \times L^2 \times L^2 : \text{ such that } \forall \varphi \in C^\infty(\mathbb{T}^3) \\ \int_{\mathbb{T}^3} \bigg(\frac{\Xi_A - \Xi_A^{j-1}}{h} \bigg) \varphi\, dx = - \int_{\mathbb{T}^3} \bigg(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{j-1}) v_i \bigg) \partial_\alpha \varphi\, dx \Bigg\}.$$

(P 2) For each $j \geqslant 1$ the iterates satisfy

$$\begin{aligned} \frac{v_i^j - v_i^{j-1}}{h} &= \partial_\alpha \bigg(\frac{\partial G}{\partial \Xi_A}(\Xi^j)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{j-1}) \bigg) \\ \frac{\Xi_A^j - \Xi_A^{j-1}}{h} &= \partial_\alpha \bigg(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{j-1})\, v_i^j \bigg) \end{aligned} \quad \text{in } \mathcal{D}'(\mathbb{T}^3). \tag{13}$$

(P 3) If F^0 is a gradient, then so is F^j for all $j \geqslant 1$.

(P 4) Iterates v^j, $j \geqslant 1$ have higher regularity: $v^j \in W^{1,p}(\mathbb{T}^3)$ for all $j \geqslant 1$.

(P 5) There exists $E_0 > 0$ determined by the initial data such that

$$\sup_{j\geqslant 0}\Big(\|v^j\|^2_{L^2_{dx}} + \int_{\mathbb{T}^3} G(\Xi^j)\,dx\Big) + \sum_{j=1}^{\infty}\|\Theta^j - \Theta^{j-1}\|^2_{L^2_{dx}} \leqslant E_0. \tag{14}$$

Given the sequence of spatial iterates (v^j, Ξ^j), $j \geqslant 1$ we define (following [9]) the time-continuous, piecewise linear interpolates $\Theta^{(h)} := (V^{(h)}, \Xi^{(h)})$ by

$$\begin{aligned} V^{(h)}(t) &= \sum_{j=1}^{\infty}\mathcal{X}^j(t)\Big(v^{j-1} + \frac{t-h(j-1)}{h}(v^j - v^{j-1})\Big)\\ \Xi^{(h)}(t) &= \big(F^{(h)}, Z^{(h)}, w^{(h)}\big)(t)\\ &= \sum_{j=1}^{\infty}\mathcal{X}^j(t)\Big(\Xi^{j-1} + \frac{t-h(j-1)}{h}(\Xi^j - \Xi^{j-1})\Big), \end{aligned} \tag{15}$$

and the piecewise constant interpolates $\theta^{(h)} := (v^{(h)}, \xi^{(h)})$ and $\tilde{f}^{(h)}$ by

$$\begin{aligned} v^{(h)}(t) &= \sum_{j=1}^{\infty}\mathcal{X}^j(t)v^j\\ \xi^{(h)}(t) &= (f^{(h)}, z^{(h)}, \omega^{(h)})(t) = \sum_{j=1}^{\infty}\mathcal{X}^j(t)\Xi^j\\ \tilde{f}^{(h)}(t) &= \sum_{j=1}^{\infty}\mathcal{X}^j(t)F^{j-1}, \end{aligned} \tag{16}$$

where $\mathcal{X}^j(t)$ is the characteristic function of the interval $I_j := [(j-1)h, jh)$. Notice that $\tilde{f}^{(h)}$ is the time-shifted version of $f^{(h)}$ and it is used later in defining a relative entropy flux, as well as the time-continuous equations (24).

Our main objective is to prove convergence of the interpolates $(V^{(h)}, F^{(h)})$ obtained via the variational scheme to the solution of polyconvex elastodynamics as long as the limit solution remains smooth. This is achieved by employing the extended system (11) and proving convergence of the time-continuous approximates $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$ to the solution $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ of the extension (11) as long as $\bar{\Theta}$ remains smooth.

Main Theorem. *Let W be defined by* (4) *with G satisfying* (H1)–(H5)*. Let $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$, $\theta^{(h)} = (v^{(h)}, \xi^{(h)})$ and $\tilde{f}^{(h)}$ be the interpolates defined via* (15)*,* (16) *and induced by the sequence of spatial iterates*

$$\Theta^j = (v^j, \Xi^j) = (v^j, F^j, Z^j, w^j) \in L^2 \times L^p \times L^2 \times L^2, \quad j \geqslant 0$$

which satisfy (P1)–(P5)*. Let $\bar{\Theta} = (\bar{V}, \bar{\Xi}) = (\bar{V}, \bar{F}, \bar{Z}, \bar{w})$ be the smooth solution of* (11) *defined on $\mathbb{T}^3 \times [0, T]$ and emanate from the data $\bar{\Theta}^0 = (\bar{V}^0, \bar{F}^0, \bar{Z}^0, \bar{w}^0)$. Assume also that $F^0, \bar{F}^0$ are gradients. Then:*

(a) *The relative entropy* $\eta^r = \eta^r(\Theta^{(h)}, \bar{\Theta})$ *defined by* (17) *satisfies* (19). *Furthermore, there exist constants* $\mu, \mu' > 0$ *such that*

$$\mu\,\mathcal{E}(t) \leqslant \int_{\mathbb{T}^3} \eta^r(x,t)\,dx \leqslant \mu'\mathcal{E}(t), \quad t \in [0,T]$$

where

$$\mathcal{E}(t) := \int_{\mathbb{T}^3} \Big((1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2})|F^{(h)} - \bar{F}|^2 + |\Theta^{(h)} - \bar{\Theta}|^2 \Big)\,dx.$$

(b) *There exists* $\varepsilon > 0$ *and* $C = C(T, \bar{\Theta}, E_0, \mu, \mu', \varepsilon) > 0$ *such that for all* $h \in (0, \varepsilon)$

$$\mathcal{E}(\tau) \leqslant C\,\big(\mathcal{E}(0) + h\big), \quad \tau \in [0,T].$$

Moreover, if the data satisfy $\mathcal{E}^{(h)}(0) \to 0$ *as* $h \downarrow 0$, *then*

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^3} \big(|\Theta^{(h)} - \bar{\Theta}|^2 + |F^{(h)} - \bar{F}|^2 (1 + |F^{(h)}|^{p-2} + |\bar{F}|^{p-2}) \big)\,dx \to 0$$

as $h \downarrow 0$.

Corollary. *Let* $\Theta^{(h)} = (V^{(h)}, \Xi^{(h)})$ *be as in the Main Theorem. Let* $(\bar{V}, \bar{F})$ *be a smooth solution of* (8) *with* $\bar{F}(\cdot, 0)$ *a gradient and* $\bar{\Theta} = (\bar{V}, \Phi(\bar{F}))$. *Assume that initial data satisfy* $\Theta^{(h)}(\cdot, 0) = \bar{\Theta}(\cdot, 0)$. *Then*

$$\sup_{t\in[0,T]} \Big(\|V - \bar{V}\|^2_{L^2(\mathbb{T}^3)} + \|\Xi^{(h)} - \Phi(\bar{F})\|^2_{L^2(\mathbb{T}^3)} + \|F^{(h)} - \bar{F}\|^p_{L^p(\mathbb{T}^3)} \Big) = O(h).$$

Remark 2.1. The smooth solution $\bar{\Theta} = (\bar{V}, \bar{\Xi})$ to the extended system (2) is provided beforehand. A natural question arises whether such a solution exists. We briefly discuss the existence theory for (1) on the torus $\mathbb{T}^3$. In [6] energy methods are used to establish local (in time) existence of smooth solutions to certain initial-boundary value problem that apply to the system of nonlinear elastodynamics (1) with rank-1 convex stored energy. More precisely, for a bounded domain $\Omega \subset \mathbb{R}^n$ with the smooth boundary $\partial\Omega$ the authors establish ([6, Theorem 5.2]) the existence of the unique motion $y(\cdot, t)$ satisfying (1) in $\Omega \times [0,T]$ together with boundary conditions $y(x,t) = 0$ on $\partial\Omega \times [0,T]$ and initial conditions $y(\cdot, 0) = y_0$ and $y_t(\cdot, 0) = y_1$ whenever $T > 0$ is small enough and the initial data lie in a compact set. One may get a counterpart of this result for solutions on $\mathbb{T}^3$ since the methods in [6] are developed in the abstract framework: a quasi-linear partial differential equation is viewed as an abstract differential equation with initial value problem set on an interpolated scale of separable Hilbert spaces $\{H_\gamma\}_{\gamma\in[0,m]}$ with $m \geqslant 2$. To be precise, the spaces satisfy $H_\gamma = [H_0, H_m]_{\gamma/m}$ and the desired solution $u(t)$ of an abstract differential equation is assumed to be taking values in $H_m \bigcap V$, where V, a closed subspace of H_1, is designated to accommodate the boundary conditions (cf. [6, Section 2]). By choosing appropriate spaces, namely

$$H_\gamma = \big[L^2(\mathbb{T}^3), W^{m,2}(\mathbb{T}^3)\big]_{\gamma/m} \quad \text{and} \quad V = H_1 = W^{1,2}(\mathbb{T}^3),$$

and requiring strong ellipticity (cf. [6, Section 5]) for the stored energy one may apply [6, Theorem 4.1] to conclude the local existence of smooth solutions on the torus $\mathbb{T}^3$ to the system of elastodynamics (1) and hence to (1). Since strong polyconvexity implies strong ellipticity [1], the same conclusion holds for the case of polyconvex energy which is used here.

Remark 2.2. The framework for existence of measure-valued solutions for the polyconvex elasticity system (see (H1)–(H4) of [9]) and that of uniqueness of classical within the class of measure-valued solutions (see [10]) is more general than the framework used in the Main Theorem. This discrepancy is due to the relative entropy being best adapted to an L^2 setting and technical difficulties connected to the estimations of the time-step approximants of (13). Our approach, based on using the "distance" function in (35) as a substitute for the relative entropy, simplifies the estimations but limits applicability to stored energies (4), (6) with L^p-growth for F but only L^2-growth in $\operatorname{cof} F$ and $\det F$.

3. Relative entropy identity

For the rest of the sequel, we suppress the dependence on h to simplify notations and, *cf.* Main Theorem, assume:

1. $\Theta = (V, \Xi)$, $\theta = (v, \xi)$, $\tilde{f}$ are the approximates defined by (15) and (16).
2. $\bar{\Theta} = (\bar{V}, \bar{\Xi}) = (\bar{V}, \bar{F}, \bar{Z}, \bar{w})$ is a smooth solution of (11) defined on $\mathbb{T}^3 \times [0, T]$ where $T > 0$ is finite.

The goal of this section is to derive an identity for a relative energy among the two solutions. To this end, we define the relative entropy

$$\eta^r(\Theta, \bar{\Theta}) := \eta(\Theta) - \eta(\bar{\Theta}) - \nabla\eta(\bar{\Theta})(\Theta - \bar{\Theta}) \tag{17}$$

and the associated relative flux which will turn out to be

$$q^r_\alpha(\theta, \bar{\Theta}, \tilde{f}) := (v_i - \bar{V}_i)\big(G_{,A}(\xi) - G_{,A}(\bar{\Xi})\big)\, \Phi^A_{,i\alpha}(\tilde{f}), \quad \alpha = 1, 2, 3. \tag{18}$$

We now state two elementary lemmas used in our further computations. The first one extends the null-Lagrangian properties while the second one provides the rule for the divergence of the product in the non-smooth case.

Lemma 3.1 (null-Lagrangian properties)**.** *Assume $q > 2$ and $r \geqslant \frac{q}{q-2}$. Then, if $u \in W^{1,q}(\mathbb{T}^3;\mathbb{R}^3)$, $z \in W^{1,r}(\mathbb{T}^3)$, we have*

$$\left.\begin{aligned} \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\right) &= 0 \\ \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u) z\right) &= \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla u)\,\partial_\alpha z \end{aligned}\right. \quad \text{in } \mathcal{D}'(\mathbb{T}^3)$$

for each $i = 1, \dots, 3$ and $A = 1, \dots, 19$.

Lemma 3.2 (product rule)**.** *Let $q \in (1,\infty)$ and $q' = \frac{q}{q-1}$. Assume*

$$f \in W^{1,q}(\mathbb{T}^3), \;\; h \in L^{q'}(\mathbb{T}^3;\mathbb{R}^3) \quad \text{and} \quad \operatorname{div} h \in L^{q'}(\mathbb{T}^3).$$

Then $fh \in L^1(\mathbb{T}^3;\mathbb{R}^3)$, $\operatorname{div}(fh) \in L^1(\mathbb{T}^3)$ and

$$\operatorname{div}(fh) = f \operatorname{div} h + \nabla f h \quad \text{in } \mathcal{D}'(\mathbb{T}^3).$$

Lemma 3.3 (relative entropy identity)**.** *For almost all $t \in [0,T]$*

$$\partial_t \eta^r - \operatorname{div} q^r = Q - \frac{1}{h}\sum_{j=1}^{\infty} \mathcal{X}^j(t) D^j + S \quad \text{in } \mathcal{D}'(\mathbb{T}^3) \tag{19}$$

where

$$\begin{aligned} Q := {} & \partial_\alpha(G_{,A}(\bar\Xi))\big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar F)\big)\big(V_i - \bar V_i\big) \\ & + \partial_\alpha \bar V_i\big(G_{,A}(\Xi) - G_{,A}(\bar\Xi)\big)\big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar F)\big) \\ & + \partial_\alpha \bar V_i\big(G_{,A}(\Xi) - G_{,A}(\bar\Xi) - G_{,AB}(\bar\Xi)(\Xi - \bar\Xi)_B\big)\Phi^A_{,i\alpha}(\bar F) \end{aligned} \tag{20}$$

estimates the difference between the two solutions,

$$D^j := \big(\nabla\eta(\theta) - \nabla\eta(\Theta)\big)\delta\Theta^j, \tag{21}$$

where $\delta\Theta^j := \Theta^j - \Theta^{j-1}$, are the dissipative terms, and

$$\begin{aligned} S := {} & \partial_\alpha(G_{,A}(\bar\Xi))\Big[\,\Phi^A_{,i\alpha}(\bar F)\big(v_i - V_i\big) + \big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar F)\big)\big(v_i - V_i\big) \\ & + \big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big)\big(v_i - V_i\big) + \big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big)\big(V_i - \bar V_i\big)\Big] \\ & + \partial_\alpha \bar V_i\Big[\big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\Phi^A_{,i\alpha}(\bar F) \\ & + \big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big) \\ & + \big(G_{,A}(\xi) - G_{,A}(\Xi)\big)\big(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar F)\big) \\ & + \big(G_{,A}(\Xi) - G_{,A}(\bar\Xi)\big)\big(\Phi^A_{,i\alpha}(\tilde f) - \Phi^A_{,i\alpha}(F)\big)\Big] \end{aligned} \tag{22}$$

is the error term.

Proof. Notice that by (15) for almost all $t \geqslant 0$

$$\begin{aligned} \partial_t V(\cdot,t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \frac{\delta v^j}{h}, \quad \delta v^j := v^j - v^{j-1} \\ \partial_t \Xi(\cdot,t) &= \sum_{j=1}^{\infty} \mathcal{X}^j(t) \frac{\delta \Xi^j}{h}, \quad \delta \Xi^j := \Xi^j - \Xi^{j-1}. \end{aligned} \tag{23}$$

Hence by (7), (13) and (23) we obtain for almost all $t \geqslant 0$

$$\left.\begin{aligned} \partial_t V_i(\cdot,t) &= \operatorname{div}\big(g_i(\xi,\tilde f)\big) \\ \partial_t \Xi_A(\cdot,t) &= \partial_\alpha\big(\Phi^A_{,i\alpha}(\tilde f)\, v_i\big) \end{aligned}\right. \quad \text{in } \mathcal{D}'(\mathbb{T}^3). \tag{24}$$

Since $(\bar V, \bar\Xi)$ is the smooth solution of (11), using (7) we also have

$$\left.\begin{aligned} \partial_t \bar V_i &= \operatorname{div}\big(g_i(\bar\Xi,\bar F)\big) \\ \partial_t \bar\Xi_A &= \partial_\alpha\big(\Phi^A_{,i\alpha}(\bar F)\, \bar V_i\big) \end{aligned}\right. \quad \text{in } \mathbb{T}^3 \times [0,T]. \tag{25}$$

Further in the proof we will perform a series of calculations that hold for smooth functions. A technical difficulty arises, since the iterates (v^j, Ξ^j), $j \geqslant 1$ satisfying (13) are, in general, not smooth. To bypass this we employ Lemmas 3.1 and 3.2 that provide the null-Lagrangian property and product rule in the smoothness class appropriate for the approximates $\Theta = (V,\Xi)$, $\theta = (v,\xi)$, $\tilde f$.

By assumption F^0 and $\bar F^0$ are gradients. Hence using (P 3) we conclude that F^j, $j \geqslant 1$ are gradients. Furthermore, from (E1) it follows that $\bar F$ remains a gradient for all t. Thus, recalling (15), (16), we have

$$F,\ f,\ \tilde f \text{ and } \bar F \text{ are gradients for all } t \in [0,T]. \tag{26}$$

We also notice that by (5), (7), and (H4) we have for all $F^* \in \mathbb{R}^9$, $\Xi^\circ \in \mathbb{R}^{19}$

$$\begin{aligned} \big|g_{i\alpha}(\Xi^\circ, F^*)\big|^{p'} &\leqslant C_g\Big(\Big|\frac{\partial G}{\partial F_{i\alpha}}\Big|^{\frac{p}{p-1}} + |F^*|^{\frac{p}{p-1}} \Big|\frac{\partial G}{\partial Z_{k\gamma}}\Big|^{\frac{p}{p-1}} + |F^*|^{\frac{2p}{p-1}} \Big|\frac{\partial G}{\partial w}\Big|^{\frac{p}{p-1}} \Big) \\ &\leqslant C_g'\Big(|F^*|^p + \Big|\frac{\partial G}{\partial F_{i\alpha}}\Big|^{\frac{p}{p-1}} + \Big|\frac{\partial G}{\partial Z_{k\gamma}}\Big|^{\frac{p}{p-2}} + \Big|\frac{\partial G}{\partial w}\Big|^{\frac{p}{p-3}} \Big) \\ &\leqslant C_g''\Big(|F^*|^p + |F^\circ|^p + |Z^\circ|^2 + |w^\circ|^2 + 1 \Big) \end{aligned} \tag{27}$$

where $p \in [6,\infty)$ and $p' = \frac{p}{p-1}$. Hence (H2), (P4)–(P5), $(16)_1$ and Lemmas 3.1, 3.2 along with $(24)_1$ imply

$$\begin{aligned} \operatorname{div}\big(v_i g_i(\xi,\tilde f)\big) &= v_i \partial_t V_i + \nabla v_i g_i(\xi,\tilde f) \\ \operatorname{div}\big(\bar V_i g_i(\xi,\tilde f)\big) &= \bar V_i \partial_t V_i + \nabla \bar V_i g_i(\xi,\tilde f) \\ \operatorname{div}\big(v_i g_i(\bar\Xi,\tilde f)\big) &= v_i \Phi^A_{,i\alpha}(\tilde f)\, \partial_\alpha(G_{,A}(\bar\Xi)) + \nabla v_i g_i(\bar\Xi,\tilde f) \\ \operatorname{div}\big(\bar V_i g_i(\bar\Xi,\tilde f)\big) &= \bar V_i \Phi^A_{,i\alpha}(\tilde f)\, \partial_\alpha(G_{,A}(\bar\Xi)) + \nabla \bar V_i g_i(\bar\Xi,\tilde f). \end{aligned} \tag{28}$$

Similarly, by (P4), Lemma 3.1, $(24)_2$ and (26) we have the identity

$$\partial_t \Xi_A(t) = \Phi^A_{,i\alpha}(\tilde{f})\, \partial_\alpha v_i. \tag{29}$$

Thus, using (12), $(28)_1$ and (29), we compute

$$\begin{aligned}
\partial_t\big(\eta(\Theta)\big) &= V_i \partial_t V_i + G_{,A}(\Xi)\partial_t \Xi_A \\
&= (V_i - v_i)\partial_t V_i + (G_{,A}(\Xi) - G_{,A}(\xi))\partial_t \Xi_A + \operatorname{div}\big(v_i g_i(\xi, \tilde{f})\big) \\
&= \frac{1}{h}\sum_{j=1}^{\infty} \mathcal{X}^j(t)\big(\nabla\eta(\Theta) - \nabla\eta(\theta)\big)\delta\Theta^j + \operatorname{div}\big(v_i g_i(\xi, \tilde{f})\big).
\end{aligned}$$

Furthermore, by $(28)_2$ we have $\partial_t\big(\bar{V}_i(V_i - \bar{V}_i)\big) = \partial_t \bar{V}_i (V_i - \bar{V}_i) + \bar{V}_i \partial_t V_i - \bar{V}_i \partial_t \bar{V}_i = \partial_t \bar{V}_i (V_i - \bar{V}_i) + \operatorname{div}\big(\bar{V}_i g_i(\xi, \tilde{f})\big) - \nabla \bar{V}_i g_i(\xi, \tilde{f}) - \frac{1}{2}\partial_t \bar{V}^2$ while using (29) we obtain

$$\begin{aligned}
\partial_t(G_{,A}(\bar{\Xi})(\Xi - \bar{\Xi})_A) &= \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A + G_{,A}(\bar{\Xi})\partial_t \Xi_A - \partial_t(G(\bar{\Xi})) \\
&= \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A + \nabla v_i g_i(\bar{\Xi}, \tilde{f}) - \partial_t(G(\bar{\Xi})).
\end{aligned}$$

Next, notice that by (7) and (18) we have

$$q^r = v_i g_i(\xi, \tilde{f}) - \bar{V}_i g_i(\xi, \tilde{f}) - v_i g_i(\bar{\Xi}, \tilde{f}) + \bar{V}_i g_i(\bar{\Xi}, \tilde{f}). \tag{30}$$

Hence by (12), (17), (21), (28) and the last four identities we obtain

$$\partial\eta^r - \operatorname{div} q^r = -\frac{1}{h}\sum_{j=1}^{\infty} \mathcal{X}^j(t) D^j + J \tag{31}$$

where

$$\begin{aligned}
J :=& -\operatorname{div}\big(\bar{V}_i g_i(\bar{\Xi}, \tilde{f})\big) + \nabla \bar{V}_i g_i(\xi, \tilde{f}) + \operatorname{div}\big(v_i g_i(\bar{\Xi}, \tilde{f})\big) - \nabla v_i g_i(\bar{\Xi}, \tilde{f}) \\
& - \partial_t \bar{V}_i (V_i - \bar{V}_i) - \partial_t(G_{,A}(\bar{\Xi}))(\Xi - \bar{\Xi})_A.
\end{aligned}$$

Consider now the term J. From (25), (26) and Lemma 3.1 it follows that $\partial_t \bar{V}_i = \Phi^A_{,i\alpha}(\bar{F})\partial_\alpha(G_{,A}(\bar{\Xi}))$, $\partial_t(G_{,A}(\bar{\Xi})) = G_{,AB}(\bar{\Xi})\Phi^B_{,i\alpha}(\bar{F})\partial_\alpha \bar{V}_i$. Then, $(28)_{3,4}$ along with the last two identities and the fact that $G_{,AB} = G_{,BA}$ implies

$$\begin{aligned}
J &= \partial_\alpha \bar{V}_i \Big(g_{i\alpha}(\xi, \tilde{f}) - g_{i\alpha}(\bar{\Xi}, \tilde{f})\Big) \\
&\quad + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(\tilde{f})(v_i - \bar{V}_i) - \Phi^A_{,i\alpha}(\bar{F})(V_i - \bar{V}_i)\Big) \\
&\quad - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_A \Phi^B_{,i\alpha}(\bar{F})\, \partial_\alpha \bar{V}_i \\
&= \partial_\alpha \bar{V}_i \Big(g_{i\alpha}(\xi, \tilde{f}) - g_{i\alpha}(\bar{\Xi}, \tilde{f}) - g_{i\alpha}(\Xi, \bar{F}) + g_{i\alpha}(\bar{\Xi}, \bar{F})\Big) \\
&\quad + \partial_\alpha(G_{,A}(\bar{\Xi}))\Big(\Phi^A_{,i\alpha}(\tilde{f})(v_i - \bar{V}_i) - \Phi^A_{,i\alpha}(\bar{F})(V_i - \bar{V}_i)\Big) \\
&\quad + \partial_\alpha \bar{V}_i \Big(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})(\Xi - \bar{\Xi})_B\Big)\Phi^A_{,i\alpha}(\bar{F}) \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{32}$$

Using (7) we rearrange the term J_1 as follows:

$$\begin{aligned}J_1 &= \partial_\alpha \bar{V}_i\Big[\big(G_{,A}(\xi)-G_{,A}(\bar{\Xi})\big)\Phi^A_{,i\alpha}(\tilde{f})-\big(G_{,A}(\Xi)-G_{,A}(\bar{\Xi})\big)\Phi^A_{,i\alpha}(\bar{F})\Big]\\ &= \partial_\alpha \bar{V}_i\Big[\big(G_{,A}(\xi)-G_{,A}(\Xi)\big)\big(\Phi^A_{,i\alpha}(\tilde{f})-\Phi^A_{,i\alpha}(F)\big)\\ &\quad+\big(G_{,A}(\xi)-G_{,A}(\Xi)\big)\big(\Phi^A_{,i\alpha}(F)-\Phi^A_{,i\alpha}(\bar{F})\big)+\big(G_{,A}(\xi)-G_{,A}(\Xi)\big)\Phi^A_{,i\alpha}(\bar{F})\\ &\quad+\big(G_{,A}(\Xi)-G_{,A}(\bar{\Xi})\big)\big(\Phi^A_{,i\alpha}(\tilde{f})-\Phi^A_{,i\alpha}(F)\big)\\ &\quad+\big(G_{,A}(\Xi)-G_{,A}(\bar{\Xi})\big)\big(\Phi^A_{,i\alpha}(F)-\Phi^A_{,i\alpha}(\bar{F})\big)\Big].\end{aligned} \tag{33}$$

We also modify the term J_2 writing it in the following way:

$$\begin{aligned}J_2 &= \partial_\alpha(G_{,A}(\bar{\Xi}))\Big[\Phi^A_{,i\alpha}(\tilde{f})(v_i-\bar{V}_i)-\Phi^A_{,i\alpha}(\bar{F})(V_i-\bar{V}_i)\Big]\\ &= \partial_\alpha(G_{,A}(\bar{\Xi}))\Big[\big(\Phi^A_{,i\alpha}(F)-\Phi^A_{,i\alpha}(\bar{F})\big)\big(V_i-\bar{V}_i\big)\\ &\quad+\big(\Phi^A_{,i\alpha}(\tilde{f})-\Phi^A_{,i\alpha}(F)\big)\big(V_i-\bar{V}_i\big)+\big(\Phi^A_{,i\alpha}(\tilde{f})-\Phi^A_{,i\alpha}(F)\big)\big(v_i-V_i\big)\\ &\quad+\big(\Phi^A_{,i\alpha}(F)-\Phi^A_{,i\alpha}(\bar{F})\big)\big(v_i-V_i\big)+\Phi^A_{,i\alpha}(\bar{F})\big(v_i-V_i\big)\Big].\end{aligned} \tag{34}$$

By (32)–(34) we have $J = J_1+J_2+J_3 = Q+S$. Hence by (31) we get (19). □

4. Proof of the Main Theorem

The identity (19) is central to our paper. In this section, we estimate each of its terms and complete the proof via Gronwall's inequality.

4.1. A function $d(\cdot,\cdot)$ equivalent to the relative entropy.

Definition. Let $\Theta_1 = (V_1,\Xi_1), \Theta_2 = (V_2,\Xi_2) \in \mathbb{R}^{22}$. We set

$$d(\Theta_1,\Theta_2) = \big(1+|F_1|^{p-2}+|F_2|^{p-2}\big)\,|F_1-F_2|^2+|\Theta_1-\Theta_2|^2 \tag{35}$$

where $(F_1,Z_1,w_1) = \Xi_1, (F_2,Z_2,w_2) = \Xi_2 \in \mathbb{R}^{19}$.

The goal of this section is to show that the relative entropy η^r can be equivalently represented by the function $d(\cdot,\cdot)$. Before we establish this relation, we prove an elementary lemma used in our further calculations:

Lemma 4.1. *Assume $q \geqslant 1$. Then for all $u, v \in \mathbb{R}^n$ and $\bar{\beta} \in [0,1]$*

$$\int_0^{\bar{\beta}}\int_0^1 (1-\beta)\,|u+\alpha(1-\beta)(v-u)|^q\,d\alpha\,d\beta \;\geqslant\; c'\bar{\beta}\big(|u|^q+|v|^q\big) \tag{36}$$

with constant $c' > 0$ depending only on q and n.

Proof. Observe first that

$$\int_0^1 |u+\alpha(v-u)|\,d\alpha \;\geqslant\; \bar{c}\,(|u|+|v|)\,, \quad \forall u,v\in\mathbb{R}^n \tag{37}$$

with $\bar{c}=\frac{1}{4\sqrt{n}}$. Then, applying Jensen's inequality and using (37), we get

$$\begin{aligned}
&\int_0^{\bar\beta}\int_0^1 (1-\beta)\big|u+\alpha(1-\beta)(v-u)\big|^q d\alpha\,d\beta\\
&\geqslant \int_0^{\bar\beta}(1-\beta)\bigg(\int_0^1 \big|u+\alpha\big((1-\beta)v+\beta u-u\big)\big|\,d\alpha\bigg)^q d\beta\\
&\geqslant \bar{c}^q\int_0^{\bar\beta}(1-\beta)\big(|u|+|(1-\beta)v+\beta u|\big)^q d\beta\\
&\geqslant \frac{\bar{c}^q}{2}\big(|u|^q+|v|^q\big)\int_0^{\bar\beta}(1-\beta)^{q+1}\,d\beta.
\end{aligned}$$

Since $q\geqslant 1$ and $(1-\bar\beta)\in[0,1]$, we have $\int_0^{\bar\beta}(1-\beta)^{q+1}d\beta = \frac{1-(1-\bar\beta)^{q+2}}{q+2} \geqslant \frac{\bar\beta}{q+2}$. Combining the last two inequalities we obtain (36). □

Lemma 4.2 (η^r-equivalence)**.** *There exist constants $\mu,\mu'>0$ such that*

$$\mu\, d(\Theta_1,\Theta_2)\;\leqslant\;\eta^r(\Theta_1,\Theta_2)\;\leqslant\;\mu' d(\Theta_1,\Theta_2) \tag{38}$$

for every $\Theta_1=(V_1,\Xi_1),\Theta_2=(V_2,\Xi_2)\in\mathbb{R}^{22}$.

Proof. Notice that

$$\begin{aligned}
\eta^r(\Theta_1,\Theta_2) &= \eta(\Theta_1)-\eta(\Theta_2)-\nabla\eta(\Theta_2)(\Theta_1-\Theta_2)\\
&= \int_0^1\int_0^1 s(\Theta_1-\Theta_2)^T\big(\nabla^2\eta(\hat\Theta)\big)(\Theta_1-\Theta_2)\,ds\,d\tau.
\end{aligned} \tag{39}$$

where $\hat\Theta=(\hat V,\hat\Xi)=(\hat V,\hat F,\hat Z,\hat w):=\Theta_2+\tau s(\Theta_1-\Theta_2)$, $\tau,s\in[0,1]$. Observe next that

$$\nabla_\Xi G = \begin{bmatrix}\nabla_F H & 0 & 0\end{bmatrix} + \nabla_\Xi R \tag{40}$$

and therefore by (12)

$$\begin{aligned}
&(\Theta_1-\Theta_2)^T\nabla^2\eta(\hat\Theta)(\Theta_1-\Theta_2)\\
&=|V_1-V_2|^2+(\Xi_1-\Xi_2)^T\nabla^2 R(\hat\Xi)(\Xi_1-\Xi_2)+(F_1-F_2)^T\nabla^2 H(\hat F)(F_1-F_2).
\end{aligned} \tag{41}$$

Then (H1), (39) and (41) imply

$$\begin{aligned}
&\tfrac12|V_1-V_2|^2+\tfrac{\gamma}{2}|\Xi_1-\Xi_2|^2+\kappa\,|F_1-F_2|^2\int_0^1\int_0^1 s|\hat F|^{p-2}ds\,d\tau\\
&\qquad\leqslant\;\eta^r(\Theta_1,\Theta_2)\;\leqslant\\
&\tfrac12|V_1-V_2|^2+\tfrac{\gamma'}{2}|\Xi_1-\Xi_2|^2+\kappa'\,|F_1-F_2|^2\int_0^1\int_0^1 s|\hat F|^{p-2}ds\,d\tau.
\end{aligned} \tag{42}$$

We now consider the integral term in (42). Recall that $\hat{F} = F_2 + \tau s(F_1 - F_2)$. Then, estimating from above, we get

$$\int_0^1 \int_0^1 s|\hat{F}|^{p-2} ds\, d\tau \leqslant 2^{p-3} \left(|F_1|^{p-2} + |F_2|^{p-2}\right)$$

while for the estimate from below we use Lemma 4.1 (with $s = 1-\beta$ and $\bar{\beta} = 1$) and obtain

$$\int_0^1 \int_0^1 s|\hat{F}|^{p-2} ds\, d\tau \geqslant c' \left(|F_1|^{p-2} + |F_2|^{p-2}\right).$$

Combining (42) with the two last inequalities we obtain (38). □

Observe that the smoothness of $\bar{\Theta}$ implies that there exists $M = M(T) > 0$ such that

$$M \geqslant |\bar{\Theta}| + |\nabla_x \bar{\Theta}| + |\partial_t \bar{\Theta}|, \quad (x,t) \in \mathbb{T}^3 \times [0,T]. \tag{43}$$

Lemma 4.3 ($\mathcal{E}$-equivalence)**.** *The relative entropy η^r and function d satisfy*

$$\eta^r(\Theta, \bar{\Theta}),\ d(\Theta, \bar{\Theta}) \in L^\infty\left([0,T]; L^1\right).$$

Moreover,

$$\mu\,\mathcal{E}(t) \leqslant \int_{\mathbb{T}^3} \eta^r\big(\Theta(x,t), \bar{\Theta}(x,t)\big)\, dx \leqslant \mu'\mathcal{E}(t), \quad t \in [0,T]$$

where

$$\mathcal{E}(t) := \int_{\mathbb{T}^3} d\big(\Theta(x,t), \bar{\Theta}(x,t)\big)\, dx$$

and constants $\mu, \mu' > 0$ are defined in Lemma 4.2.

Proof. Fix $t \in [0,T]$. Then there exists $j \geqslant 1$ such that $t \in I_j$. Hence (15), (35), (43) and (H2) imply for $p \in [6, \infty)$

$$\begin{aligned} d(\Theta(\cdot,t), \bar{\Theta}(\cdot,t)) &\leqslant C\Big(1 + |F|^p + |Z|^2 + |w|^2 + |V|^2\Big) \\ &\leqslant C\Big(1 + G(\Xi^{j-1}) + G(\Xi^j) + |v^{j-1}|^2 + |v^j|^2\Big) \end{aligned} \tag{44}$$

with $C = C(M) > 0$ independent of h, j and t. Hence (14) and (44) imply

$$\int_{\mathbb{T}^3} d(\Theta(\cdot,t), \bar{\Theta}(\cdot,t))\, dx \leqslant C'(1 + E_0), \quad \forall t \in [0,T] \tag{45}$$

for some $C' = C'(M) > 0$. Then (38) and (45) imply the lemma. □

4.2. Estimate for the term Q on $t \in [0,T]$.

Lemma 4.4 (Q-bound)**.** *There exists $\lambda = \lambda(M) > 0$ such that*

$$|Q(x,t)| \leqslant \lambda\, d(\Theta, \bar{\Theta}), \quad (x,t) \in \mathbb{T}^3 \times [0,T] \tag{46}$$

where the term Q is defined by (20)*.*

Proof. Let $C = C(M) > 0$ be a generic constant. Notice that for all $F_1, F_2 \in M^{3\times 3}$

$$\big|\Phi^A_{,i\alpha}(F_1) - \Phi^A_{,i\alpha}(F_2)\big| \leqslant \begin{cases} 0, & A = 1,\dots,9 \\ |F_1 - F_2|, & A = 10,\dots,18 \\ 3\big(|F_1| + |F_2|\big)|F_1 - F_2|, & A = 19 \end{cases} \tag{47}$$

and hence

$$|\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F})| \leqslant C\,(1 + |F|)\,\big|F - \bar{F}\big|, \quad A = 1,\dots 19. \tag{48}$$

Then, using (43) and (48) we estimate the first term of Q:

$$\big|\partial_\alpha(G_{,A}(\bar{\Xi}))(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F}))(V_i - \bar{V}_i)\big| \leqslant C\big((1+|F|^2)|F-\bar{F}|^2 + |V-\bar{V}|^2\big). \tag{49}$$

Observe now that (40) and $(47)_1$ imply for all $\Xi_1, \Xi_2 \in \mathbb{R}^{22}$, $F_3, F_4 \in \mathbb{R}^9$

$$\begin{aligned} &(G_{,A}(\Xi_1) - G_{,A}(\Xi_2))(\Phi^A_{,i\alpha}(F_3) - \Phi^A_{,i\alpha}(F_4)) \\ &= (R_{,A}(\Xi_1) - R_{,A}(\Xi_2))(\Phi^A_{,i\alpha}(F_3) - \Phi^A_{,i\alpha}(F_4)). \end{aligned} \tag{50}$$

Thus, by (H1), (48) and (50) we obtain the estimate for the second term:

$$\big|\partial_\alpha \bar{V}_i (G_{,A}(\Xi) - G_{,A}(\bar{\Xi}))(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F}))\big| \leqslant C\big(|\Xi - \bar{\Xi}|^2 + (1+|F|^2)|F - \bar{F}|^2\big). \tag{51}$$

Finally, we define for each $A = 1,\dots,19$

$$\begin{aligned} J_A &:= G_{,A}(\Xi) - G_{,A}(\bar{\Xi}) - G_{,AB}(\bar{\Xi})\,\big(\Xi - \bar{\Xi}\big)_B \\ &= \int_0^1\!\!\int_0^1 s(\Xi - \bar{\Xi})^T \nabla^2 G_{,A}(\hat{\Xi})(\Xi - \bar{\Xi})\, ds\, d\tau \end{aligned} \tag{52}$$

where $\hat{\Xi} = (\hat{F}, \hat{Z}, \hat{w}) := \bar{\Xi} + \tau s(\Xi - \bar{\Xi})$, $\tau, s \in [0,1]$. By (6) and (H5) we have for each $A = 1,\dots,19$

$$\big|(\Xi - \bar{\Xi})^T \nabla^2 G_{,A}(\hat{\Xi})(\Xi - \bar{\Xi})\big| \leqslant C\big(|F - \bar{F}|^2 |\hat{F}|^{p-3} + |\Xi - \bar{\Xi}|^2\big). \tag{53}$$

Then by (43) and (52), (53) we obtain the estimate for the third term:

$$\begin{aligned} |\partial_\alpha \bar{V}_i\, \Phi^A_{,i\alpha}(\bar{F})\, J_A| &\leqslant C\Big(|\Xi - \bar{\Xi}|^2 + |F - \bar{F}|^2 \int_0^1\!\!\int_0^1 |\bar{F} + \tau s(F - \bar{F})|^{p-3} ds\, d\tau\Big) \\ &\leqslant C\Big(|\Xi - \bar{\Xi}|^2 + |F - \bar{F}|^2(1 + |F|^{p-3})\Big). \end{aligned} \tag{54}$$

Thus by (35), (49), (51) and (54) we conclude for $p \in [6, \infty)$

$$|Q(x,t)| \leqslant C\Big(|\Theta - \bar{\Theta}|^2 + (1 + |F|^{p-2})|F - \bar{F}|^2\Big) \leqslant C\, d(\Theta, \bar{\Theta}). \qquad \square$$

4.3. Estimates for the terms D^j and S on $t \in I_j' \subset [0,T]$. In this section, we consider $j \geqslant 1$ such that $(j-1)h < T$ and estimate the dissipative and error terms for $t \in I_j'$ where

$$I_j' := I_j \bigcap [0,T] = [(j-1)h, jh) \bigcap [0,T].$$

Lemma 4.5 (D^j-bound)**.** *Let D^j be the term defined by* (21)*. Then*

$$D^j \in L^\infty\big(I_j'; L^1(\mathbb{T}^3)\big) \tag{55}$$

and there exists constant $C_D > 0$ independent of h and j such that for all times $\tau \in \bar{I}_j' := [(j-1)h, jh] \bigcap [0,T]$

$$\int_{(j-1)h}^{\tau} \int_{\mathbb{T}^3} \Big(\frac{1}{h} D^j\Big)\, dx\, dt \geqslant a(\tau) C_D \int_{\mathbb{T}^3} |\delta\Theta^j|^2 + \big(|F^{j-1}|^{p-2} + |F^j|^{p-2}\big)|\delta F^j|^2\, dx \geqslant 0 \tag{56}$$

with

$$a(\tau) := \frac{\tau - h(j-1)}{h} \in [0,1], \quad \tau \in \bar{I}_j'. \tag{57}$$

Proof. By (H1), (12) and the definition of D^j we have for $t \in I_j'$

$$D^j = (v - V)\,\delta v^j + \big(\nabla H(f) - \nabla H(F)\big)\delta F^j + \big(\nabla R(\xi) - \nabla R(\Xi)\big)\delta \Xi^j. \tag{58}$$

Consider each of the three terms in (58). Notice that, by (15), (16), we have

$$\begin{aligned} v(\cdot,t) - V(\cdot,t) &= (1 - a(t))\,\delta v^j \\ \xi(\cdot,t) - \Xi(\cdot,t) &= (1 - a(t))\,\delta \Xi^j. \end{aligned} \tag{59}$$

Using (59) we compute

$$\begin{aligned} \big(v - V\big)\delta v^j &= (1 - a(t))\,|\delta v^j|^2 \\ \big(\nabla R(\xi) - \nabla R(\Xi)\big)\delta\Xi^j &= (1 - a(t)) \int_0^1 (\delta\Xi^j)^T \nabla^2 R(\hat{\Xi})\,(\delta\Xi^j)\,ds \\ \big(\nabla H(f) - \nabla H(F)\big)\delta F^j &= (1 - a(t)) \int_0^1 (\delta F^j)^T \nabla^2 H(\hat{F})\,(\delta F^j)\,ds \end{aligned} \tag{60}$$

where $\hat{\Xi} = (\hat{F}, \hat{Z}, \hat{w}) := s\xi(\cdot,t) + (1-s)\Xi(\cdot,t)$, $s \in [0,1]$. Then (H1), (58) and (60) together with the fact that $(1 - a(t)) \in [0,1]$ imply

$$\big|D^j(\cdot,t)\big| \leqslant \Big(|\delta v^j|^2 + \gamma'|\delta\Xi^j|^2 + \kappa'|\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds\Big). \tag{61}$$

Consider now the two latter terms in (61). Recalling that $\hat{F} = sf - (1-s)F$ and using (H2) together with (15), (16) we obtain

$$\begin{aligned}\gamma'|\delta\Xi^j|^2 + \kappa'|\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds \\ \leqslant C\left(1 + |F^{j-1}|^p + |F^j|^p + |Z^{j-1}|^2 + |Z^j|^2 + |w^{j-1}|^2 + |w^j|\right)\end{aligned}$$

for some $C > 0$ independent of h, j and t. Thus, combining the last inequality with (H2), the growth estimate (14) and (61), we conclude

$$\int_{\mathbb{T}^3} \left|D^j(x,t)\right| \, dx \leqslant \nu'\bigl(1 + E_0\bigr), \quad \forall t \in I'_j$$

for some $\nu' > 0$ independent of h, j and t. This proves (55).

Let us now estimate D^j from below. By (58), (60) and (H1) we obtain

$$D^j(\cdot,t) \geqslant \nu\,(1 - a(t))\left(|\delta\Theta^j|^2 + |\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds\right) \geqslant 0 \qquad (62)$$

for $\nu = \min(1,\gamma,\kappa) > 0$. Notice that

$$\hat{F}(s,t) = sf(t) + (1-s)F(t) = F^j + (1-s)(1-a(t))(F^{j-1} - F^j).$$

Then, by making use of Lemma 4.1 we obtain for $\tau \in \bar{I}'_j$

$$\begin{aligned}&\int_{(j-1)h}^{\tau} \left((1-a(t))\, |\delta F^j|^2 \int_0^1 |\hat{F}(s,t)|^{p-2} ds\right) dt \\ &= h|\delta F^j|^2 \int_0^{a(\tau)} \int_0^1 (1-\beta)|F^j + \alpha(1-\beta)(F^{j-1} - F^j)|^{p-2} d\alpha\, d\beta \\ &\geqslant h\, a(\tau)\, c'\bigl(|F^{j-1}|^{p-2} + |F^j|^{p-2}\bigr)|\delta F^j|^2\end{aligned}$$

where we used the change of variables $\alpha = 1 - s$ and $\beta = a(t)$. Similarly, we get

$$\int_{(j-1)h}^{\tau} (1-a(t))\, |\delta\Theta^j|^2 \, dt = h\, |\delta\Theta^j|^2 \int_0^{a(\tau)} (1-\beta)\, d\beta \geqslant \frac{h\, a(\tau)}{2} |\delta\Theta^j|^2.$$

Then (62) and the last two estimates imply (56) for $C_D = \min(\nu c', \frac{\nu}{2}) > 0$. □

Lemma 4.6 (S-bound)**.** *Let S be the term defined by* (22)*. Then*

$$S \in L^\infty\bigl(I'_j; L^1(\mathbb{T}^3)\bigr) \qquad (63)$$

and there exists constant s$C_S > 0$ independent of h, j such that for any $\varepsilon > 0$ and all $\tau \in \bar{I}'_j$

$$\begin{aligned}&\int_{(j-1)h}^{\tau}\int_{\mathbb{T}^3} |S(x,t)|\,dx\,dt\\ &\leqslant C_S\Big[a(\tau)(h+\varepsilon)\int_{\mathbb{T}^3}|\delta\Theta^j|^2 + (|F^{j-1}|^{p-2}+|F^j|^{p-2})|\delta F^j|^2\,dx\\ &\quad + \frac{a(\tau)h^2}{\varepsilon}(3+2E_0) + \int_{(j-1)h}^{\tau}\int_{\mathbb{T}^3} d(\Theta,\bar{\Theta})\,dx\,dt\Big]\end{aligned} \tag{64}$$

with $a(\tau)$ defined by (57).

Proof. As before, we let $C = C(M) > 0$ be a generic constant and remind the reader that all estimates are done for $t \in I'_j$.

Observe that $(15)_2$, $(16)_3$ and (57) imply $F(\cdot,t) - \tilde{f}(\cdot,t) = a(t)\delta F^j$. Hence by $(15)_2$, $(16)_3$, (47), (57) and the identity above we get the estimate

$$\big|\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F)\big| \leqslant C\big(1+|\tilde{f}|+|F|\big)|F-\tilde{f}| \leqslant C\left(1+|F^{j-1}|+|F^j|\right)|\delta F^j|. \tag{65}$$

Thus (48), (57), $(59)_1$, (65) and Young's inequality imply

$$\begin{aligned}&\big|\Phi^A_{,i\alpha}(\bar{F})(v_i - V_i)\big| + \big|(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F}))(v_i - V_i)\big|\\ &+ \big|(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F))(v_i - V_i)\big| + \big|(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F))(V_i - \bar{V}_i)\big|\\ &\leqslant C\Big(|\delta v^j| + (1+|F|^2)|F-\bar{F}|^2 + |\delta v^j|^2 + (1+|F^{j-1}|^2+|F^j|^2)|\delta F^j|^2\\ &\quad + |V-\bar{V}|^2\Big).\end{aligned} \tag{66}$$

We also notice that for all $F_1, F_2 \in M^{3\times 3}$

$$H_{,i\alpha}(F_1) - H_{,i\alpha}(F_2) = \int_0^1 \frac{\partial^2 H}{\partial F_{i\alpha}\partial F_{lm}}\big(sF_1 + (1-s)F_2\big)(F_1 - F_2)_{lm}\,ds.$$

Hence (H1), (H5), (57), $(59)_2$ and the identity above imply

$$\begin{aligned}\big|\Phi^A_{,i\alpha}(\bar{F})\,(G_{,A}(\xi) - G_{,A}(\Xi))\big| &\leqslant C\big(|\nabla H(f) - \nabla H(F)| + |\nabla R(\xi) - \nabla R(\Xi)|\big)\\ &\leqslant C\Big(|f-F|\int_0^1 |sf+(1-s)F|^{p-2}ds + |\xi-\Xi|\Big)\\ &\leqslant C\big((|F^{j-1}|^{p-2}+|F^j|^{p-2})|\delta F^j| + |\delta\Xi^j|\big).\end{aligned} \tag{67}$$

Next, by (H1), (48), (50), (57), $(59)_2$ and (65) we obtain

$$\begin{aligned}&\big|(G_{,A}(\xi) - G_{,A}(\Xi))(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F))\big|\\ &+ \big|(G_{,A}(\xi) - G_{,A}(\Xi))(\Phi^A_{,i\alpha}(F) - \Phi^A_{,i\alpha}(\bar{F}))\big|\\ &+ \big|(G_{,A}(\Xi) - G_{,A}(\bar{\Xi}))(\Phi^A_{,i\alpha}(\tilde{f}) - \Phi^A_{,i\alpha}(F))\big|\\ &\leqslant C\Big(|\delta\Xi^j|^2 + (1+|F^{j-1}|^2+|F^j|^2)|\delta F^j|^2 + (1+|F|^2)|F-\bar{F}|^2 + |\Xi-\bar{\Xi}|^2\Big).\end{aligned} \tag{68}$$

Finally, (22), (43), and the estimates (66)-(68) imply for $p \in [6, \infty)$

$$\begin{aligned} |S(\cdot,t)| \leqslant C_S \Big[&(|F^{j-1}|^{p-2} + |F^j|^{p-2}) |\delta F^j|^2 + |\delta\Theta^j|^2 \\ &+ (|F^{j-1}|^{p-2} + |F^j|^{p-2}) |\delta F^j| + |\delta\Theta^j| + d(\Theta, \bar\Theta) \Big] \end{aligned} \tag{69}$$

for some $C_S > 0$ independent of h, j and t. Then, by (14) and (44) we conclude that the right hand side of (69) is in $L^\infty\big(I'_j; L^1(\mathbb{T}^3)\big)$ which proves (63).

We now pick any $\varepsilon > 0$. Then, employing Young's inequality, we obtain

$$(|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j| \leqslant \frac{h}{\varepsilon}\big(|F^{j-1}|^{p-2} + |F^j|^{p-2}\big) + \frac{\varepsilon}{h}\big(|F^{j-1}|^{p-2} + |F^j|^{p-2}\big)|\delta F^j|^2$$

and, similarly, $|\delta\Theta^j| \leqslant \frac{h}{\varepsilon} + \frac{\varepsilon}{h}|\delta\Theta^j|^2$. Thus (69) and the last two estimates imply

$$\begin{aligned} |S(\cdot,t)| \leqslant C_S \Big[&\big(1 + \frac{\varepsilon}{h}\big)\big(|\delta\Theta^j|^2 + (|F^{j-1}|^{p-2} + |F^j|^{p-2})|\delta F^j|^2 \big) \\ &+ \frac{h}{\varepsilon}\big(1 + |F^{j-1}|^{p-2} + |F^j|^{p-2}\big) + d(\Theta, \bar\Theta) \Big]. \end{aligned} \tag{70}$$

To this end, we integrate (70) and use (H2) along with (14) to get (64). □

4.4. Conclusion of the proof via Gronwall's inequality.

We now estimate the left hand side of the relative entropy identity (19):

Lemma 4.7 (LHS estimate). *Let η^r, q^r be the relative entropy and relative entropy flux, respectively, defined by* (17) *and* (18). *Then*

$$\Big(\partial_t \eta^r - \operatorname{div} q^r\Big) \in L^\infty\big([0,T], L^1(\mathbb{T}^3)\big) \tag{71}$$

and there exists $\bar\varepsilon > 0$ such that for all $h \in (0, \bar\varepsilon)$ and $\tau \in [0,T]$

$$\int_0^\tau \int_{\mathbb{T}^3} \big(\partial_t\, \eta^r - \operatorname{div} q^r\big)\, dx\, dt \leqslant C_I\Big(\tau h + \int_0^\tau \int_{\mathbb{T}^3} d(\Theta, \bar\Theta)\, dx\, dt\Big). \tag{72}$$

for some constant $C_I = C_I(M, E_0, \bar\varepsilon) > 0$.

Proof. Lemma 4.2, (46), (55), and (63) imply that the right hand side of the relative entropy identity (19) is in $L^\infty([0,T]; L^1(\mathbb{T}^3))$. This proves (71).

Notice that the constants C_D and C_S (that appear in Lemmas 4.5 and 4.6, respectively) are independent of h, j. Then set $\bar\varepsilon := \frac{C_D}{2C_S}$. Take now $h \in (0, \bar\varepsilon)$ and $\tau \in [0,T]$. Using Lemmas 4.4, 4.5 and 4.6 (with $\varepsilon = \bar\varepsilon$) along with the fact that $-C_D + C_S(h + \bar\varepsilon) \leqslant 0$ we get

$$\int_0^\tau \int_{\mathbb{T}^3} \Big(-\frac{1}{h} \sum_{j=1}^\infty \mathcal{X}^j(t)\, D^j + |S| + |Q| \Big) dx dt \leqslant C_I\Big(\tau h + \int_0^\tau \int_{\mathbb{T}^3} d(\Theta, \bar\Theta) dx dt\Big)$$

with $C_I := 3\max\big(C_S \frac{1+E_0}{\bar\varepsilon}, C_S + \lambda\big) > 0$. Hence by (19) and the estimate above we obtain (72). □

Observe that (P4), (P5), (18), (23), (27), (28), and (30) imply

$$\operatorname{div} q^r \in L^\infty\big([0,T]; L^1(\mathbb{T}^3)\big)$$

and hence by (71)

$$\partial_t \eta^r \in L^\infty\big([0,T]; L^1(\mathbb{T}^3)\big)\,. \tag{73}$$

Take now arbitrary $h \in (0,\bar{\varepsilon})$ and $\tau \in [0,T]$. Due to periodic boundary conditions (by the density argument) we have $\int_{\mathbb{T}^3}\big(\operatorname{div} q^r(x,s)\big)\,dx = 0$ for a.e. $s \in [0,T]$ and hence

$$\int_0^\tau \int_{\mathbb{T}^3} \operatorname{div} q^r \, dx\, dt = 0.$$

Finally, by construction for each fixed $\bar{x} \in \mathbb{T}^3$ the function $\eta^r(\bar{x},t) : [0,T] \to \mathbb{R}$ is absolutely continuous with the weak derivative $\partial_t \eta^r(\bar{x},t)$. Then, by (73) and Fubini's theorem we have

$$\int_0^\tau \int_{\mathbb{T}^3} \partial_t \eta^r dx\, dt = \int_{\mathbb{T}^3}\left[\int_0^\tau \partial_t \eta^r(x,t)\, d\tau\right] dx = \int_{\mathbb{T}^3}\Big(\eta^r(x,\tau) - \eta^r(x,0)\Big)dx.$$

Thus by Lemma 4.3, (71)-(73) and the two identities above we obtain

$$\mathcal{E}(\tau) \leqslant \bar{C}\Big(\mathcal{E}(0) + \int_0^\tau \mathcal{E}(t)\, dt + h\Big) \tag{74}$$

with $\bar{C} := \frac{T}{\mu}\max(C_I, \mu')$ independent of τ, h. Since $\tau \in [0,T]$ is arbitrary, by (74) and Gronwall's inequality we conclude

$$\mathcal{E}(\tau) \leqslant \bar{C}\big(\mathcal{E}(0) + h\big)e^{\bar{C}T}, \quad \forall \tau \in [0,T].$$

In this case, if $\mathcal{E}^{(h)}(0) \to 0$ as $h \downarrow 0$, then $\sup_{\tau\in[0,T]}\big(\mathcal{E}^{(h)}(\tau)\big) \to 0$, as $h \downarrow 0$.

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